ANALYSIS OF AXISYMMETRICALLY LOADED CYLINDRICAL AND SPHERICAL PRESSURE VESSELS FOR VARIOUS LOADING CONDITIONS

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THESIS

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ABSTRACT

Formulas relating to the elastic behavior of axisymmetrically loaded cylindrical and spherical pressure vessels have been collected from various sources. Their presentation is unified by the employment of a uniform system of notation. Loadings include interior and exterior pressurization, radial and axial temperature gradients, axial loads, and centrifugal force fields. Some cases of buckling are treated, and shrink assembled multilayer vessels are included. A large bibliography is listed.

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NOTATIONS

(Dimensions are indicated in brackets)

In the formulas which appear in this thesis, great care is exercised to assure that superscripts and exponents can not be confused. Thus, we write $n \times n = (n)^2$ rather than n^2, since the latter could be interpreted as the value of n which pertains to section two. However, the expression $1 - (\nu)^2$ occurs so frequently that we write $1 - \nu^2$, omitting the parentheses for simplicity.

ENGLISH LETTERS

b	[L]	Distance from the end of cylindrical pressure vessels to any selected point.
B	[FL^{-4}]	A parameter used only in Section XLIII.
C	[-]	Parameter = $\cosh(\beta_c x)$. $C^2 = \cosh(\beta_c \ell)$.
c	[-]	Parameter = $\cos(\beta_c s)$. $c^2 = \cos(\beta_c \ell)$.
D	[FL]	Shell flexural rigidity in general.
D_c	[FL]	Flexural rigidity of cylindrical pressure vessels $\equiv \dfrac{E(t_c)^3}{12(1-\nu_c^2)}$.
D_s	[FL]	Flexural rigidity of spherical pressure vessels $= \dfrac{E(t_s)^3}{12(1-\nu_s^2)}$.
D_p	[FL]	Flexural rigidity of plates $= \dfrac{E(t_p)^3}{12(1-\nu_p^2)}$.
d	[L]	Diameter of pressure vessels in general.
d_c	[L]	Mean diameter of cylindrical pressure vessels.

d_{ci}	[L]	Inner diameter of cylindrical pressure vessels.
d_{co}	[L]	Outer diameter of cylindrical pressure vessels.
d_s	[L]	Mean diameter of spherical pressure vessels.
d_{si}	[L]	Inner diameter of spherical pressure vessels.
d_{so}	[L]	Outer diameter of spherical pressure vessels.
E	$[FL^{-2}]$	Modulus of elasticity.
E_i	$[FL^{-2}]$	Modulus of elasticity of i^{th} cylinder of multilayer pressure vessels.
E_t	$[FL^{-2}]$	Tangent modulus of elasticity.
F	[F]	Axial tensile load on pressure vessels.
F_{cr}	[F]	Critical value for axial buckling force.
g	$[LT^{-2}]$	Gravitational acceleration.
H	$[FL^{-1}]$	Edge Shear force (per unit length) in general.
H_j	$[FL^{-1}]$	Edge shear force on pressure vessels in which the subscript j denotes the junction number where the edge shear force exists.
I	[-]	Stress index in general = $\frac{\sigma}{(\frac{pd}{2t})}$.
$I_x^{Y_j}$	[-]	Stress index in which subscript x denotes the type of pressure vessel; superscript Y denotes the type of loading, j denotes the junction number where the stress is evaluated.
k_c	$[FL^{-3}]$	Parameter for cylindrical pressure vessels $= \frac{Et_c}{(R_c)^2}$.
k_s	$[FL^{-3}]$	Parameter for spherical pressure vessels $= \frac{Et_s}{(R_s)^2}$.
ℓ	[L]	Length of cylindrical pressure vessels.
M	[F]	Edge bending moment (per unit length) in general.

M_j	[F]	Edge bending moment on pressure vessels in which subscript j denotes the junction number where the edge bending moment exists.
M_{cr}	[FL]	Critical value for torsional buckling moment on cylinder. (Note dimensions are [FL] rather than [F].)
m	[-]	Parameter = $\sqrt[4]{12(1-\nu^2)}$
p	$[FL^{-2}]$	Pressure in general.
p_{cr}	$[FL^{-2}]$	Critical value for external pressure.
p_i	$[FL^{-2}]$	Internal pressure.
p_o	$[FL^{-2}]$	External pressure.
p^{*i}_{xk}	$[FL^{-2}]$	Successive interface pressures of pressurized multilayer vessels with shrink-fit, in which subscript x denotes the type of pressure vessel, k denotes inner or outer surface, and superscript i denotes the layer number.
q	$[FL^{-1}]$	Axial tensile force per unit circumference $= \frac{F}{2\pi R_c}$
R	[L]	Radius of pressure vessels in general.
R_c	[L]	Mean radius of cylindrical pressure vessels.
R_{ci}	[L]	Inner radius of cylindrical pressure vessels.
R_{co}	[L]	Outer radius of cylindrical pressure vessels.
R_s	[L]	Mean radius of spherical pressure vessels.
R_{si}	[L]	Inner radius of spherical pressure vessels.
R_{co}	[L]	Outer radius of spherical pressure vessels.
R^i_{xk}	[L]	Radius of each pressure vessel member of multilayer pressure vessel in which subscript x denotes the type of pressure vessel, k denotes inner or outer radius, and i denotes the layer number.
r	[L]	Distance in radial direction (subscript r indicates radial direction)

S	[-]	Parameter = $\sinh(\beta_c x)$. $S^2 = \sinh(\beta_c \ell)$.
S_t	[FL^{-2}]	Equipment allowable tensile stress.
s	[-]	Parameter = $\sin(\beta_c x)$. $s^2 = \sin(\beta_c \ell)$.

In the immediately following definitions "temperature" means increase in temperature over a uniform initial temperature.

$T_{(r)}$	[°F]	Temperature, as a function of radius.
ΔT	[°F]	Temperature difference in radial direction, positive if the outside is hotter than the inside.
$\overline{T}$	[°F]	Average temperature through wall of vessel as a function of axial position.
T_i	[°F]	Temperature of inner surface.
T_o	[°F]	Temperature of outer surface, also axial difference in temperature in Section XLV.
t	[L]	Thickness of pressure vessels in general. (Subscript indicates tangential or circumferential direction, except see t_t below.)
t_c	[L]	Thickness of cylindrical pressure vessels.
t_e	[L]	Thickness of ellipsoidal pressure vessels.
t_{cn}	[L]	Thickness of conical pressure vessels.
t_p	[L]	Thickness of plate.
t_s	[L]	Thickness of spherical pressure vessels.
t_t	[L]	Thickness of toroidal pressure vessels.
u	[L]	Radially outward deformation of pressure vessels.
u_i	[L]	Deformation of pressure vessels in which subscript i denotes pressure vessel number to be taken.

x [L] Distance in axial direction. (Subscript x indicates axial direction.)

y [L] Distance in y direction. (Subscript y indicates y direction.)

GREEK LETTERS

α [(temperature)$^{-1}$] Coefficient of thermal expansion.

β_c [L^{-1}] Parameter for cylindrical pressure vessels

$$= \frac{Et_c}{4D_c(R_c)^2} = \sqrt[4]{\frac{3(1-\nu_c{}^2)}{(R_c)^2(t_c)^2}}$$

β_s [L^{-1}] Parameter for spherical pressure vessels

$$= \frac{Et_s}{4D_s(R_s)^2} = \sqrt[4]{\frac{3(1-\nu_s{}^2)}{(R_s)^2(t_s)^2}}$$

γ [FL^{-3}] Specific weight, (i.e., weight per unit volume).

δ [L] Deflection or displacement in general.

δ^* [L] Radial interference (as manufactured) prior to assembly of layers of multilayer pressure vessels.

δ_c [L] Radial deflection or displacement of cylindrical pressure vessels.

δ_{ci} [L] Radial deflection or displacement of inner surface of cylindrical pressure vessels.

δ_{co} [L] Radial deflection or displacement of outer surface of cylindrical pressure vessels.

δ_s [L] Radial deflection or displacement of spherical pressure vessels.

δ_{si} [L] Radial deflection or displacement of inner surface of spherical pressure vessels.

δ_{so} [L] Radial deflection or displacement of outer surface of spherical pressure vessels.

δ_x^i [L] Radial deflection at the edge (end) of pressure vessels in which subscript x denotes the type of pressure vessel, superscript i denotes the junction number where edge deflection occurs.

$\delta_x^{iY_j}$ [*] The edge deflection influence coefficient of pressure vessels in which the subscript x denotes the type of pressure vessel, superscript i denotes the junction number where the edge deflection coefficient is to be taken, Y denotes the type of loading, j denotes the junction number where the loading exists. [*] indicates that dimension depends on the dimension of the corresponding (Y) loading.

ε [-] Unit strain.

θ [—] Rotation of edge of pressure vessels in general. (All angles are measured in radians except when the symbol ° is added; e.g., $\theta°$.)

θ_x^i [—] Edge rotation of pressure vessels in which subscript x denotes the type of pressure vessel, superscript i denotes the junction number where the edge rotation is to be taken.

$\theta_x^{iY_j}$ [*] Edge rotation influence coefficient of pressure vessels in which the subscript x denotes the type of pressure vessel, superscript i denotes the junction number where the edge rotation coefficient is to be taken, Y denotes the type of loading, j denotes the junction number where the loading exists. [*] indicates that the dimension depends on (Y) loading.

λ_c [-] Parameter for cylindrical pressure vessels

$$= \sqrt[4]{3(1-\nu_c^2)\left(\frac{R_c}{t_c}\right)^2}$$

λ_s [-] Parameter for spherical pressure vessels

$$= \sqrt[4]{3(1-\nu_s^2)\left(\frac{R_s}{t_s}\right)^2}$$

μ [-] Parameter $= \dfrac{m}{d_c t_c} \times \dfrac{M_1}{H_1}$

ν [-] Poisson's ratio.

ν_i [-] Poisson's ratio of i^{th} cylinder of multilayer pressure vessels.

ξ [-] Parameter in Section XIX.

ρ	$[FT^2L^{-4}]$	Mass per unit volume = $\frac{\gamma}{g}$.
σ	$[FL^{-2}]$	Stress in general.
σ_{cr}	$[FL^{-2}]$	Critical value for compressive axial buckling stress of pressure vessels.
σ_{ℓ}	$[FL^{-2}]$	Meridional stress.
σ_r	$[FL^{-2}]$	Radial stress.
σ_r^i	$[FL^{-2}]$	Radial stress in i^{th} layer of multilayer vessel.
σ_t	$[FL^{-2}]$	Tangential or circumferential stress.
σ_t^i	$[FL^{-2}]$	Tangential or circumferential stress of i^{th} layer of multilayer vessel.
$(\sigma_t)^i$	$[FL^{-2}]$	Larger hoop or circumferential tension at radius R^i_{co} in multilayer pressure vessels.
$(\sigma_t)^i_s$	$[FL^{-2}]$	Smaller hoop or circumferential tension at radius R^i_{co} in multilayer pressure vessels.
σ_x	$[FL^{-2}]$	Axial stress.
σ_x^i	$[FL^{-2}]$	Axial stress in i^{th} layer of multilayer vessel.
σ_{yp}	$[FL^{-2}]$	Yield stress.
τ	$[FL^{-2}]$	Shear stress in general.
τ_{cr}	$[FL^{-2}]$	Critical value for buckling shear stress due to torsional moment.
Φ	[-]	Angle in circumferential direction measured from any selected point.
ϕ	[-]	Angle in longitudinal direction measured from a plane perpendicular to the axis of pressure vessel.

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I. INTRODUCTION

This thesis presents a collection of formulas applicable to mechanical (structural) analysis and design of axisymmetric pressure vessels under axisymmetric loadings. Emphasis is on cylindrical vessels, but some results are given for spherical vessels. Furthermore, a variety of end closures are considered in connection with long thin-walled cylindrical pressure vessels.

No claim is made for any originality of content. For the most part the results presented herein may be considered to be a part of the general or common knowledge of the mechanics of pressure vessels and they are given in many references. Generally, one or more literature citations are indicated herein pertaining to each set of results. The literature contains several instances of what can be considered as errors by authors and compilers of good reputations, but which we choose to consider as misunderstandings resulting from imperfect specification of auxiliary conditions. For this reason we have refrained from pointing out such cases in the literature but have attempted to assure that our own reporting of results is not subject to such misunderstandings.

No attempt is made herein to give derivations of results which are dealt with extensively elsewhere except that Appendices 1 and 2 hereof do provide, in what we believe to

be a very compact and useful form, basic analyses for semi-infinite and finite (respectively) thin-walled cylinders, results from which are employed extensively in the body thereof. Also, Appendix 3, contributed by Professor Brock, provides, in compact form, an analysis which, though elementary, we have not found elsewhere.

Also, no claim is made for completeness. The literature is so very extensive and is growing at such a rate that it would be quite impossible to report all significant and useful results. Time and space limitations have thus limited the scope of what is here presented. What does appear, then, reflects the interest of the author who has no wish to defend his choices or to rebut any charges of inconsistency in selection.

However, it is indeed hoped that this compendium will be found useful by those into whose hands it falls. Most, or at least a great many, of the commonly encountered loading situations are treated. An attempt has been made to be quite detailed and complete in delineating all pertinent constraints and conditions. (For example, the influence of axial strain or axial loading is explicitly considered, thus tending to avoid the errors or misconceptions which frequently arise when this matter is evaded.) In some cases adequately treated in the literature it is nevertheless difficult to obtain a clear picture of alternate modes of behavior and the circumstance under which each may occur. In these cases we hope that the broader picture may be usefully

evident herein. (One such case is that of torsional buckling of thin-walled cylinders.) Finally, the literature citations which appear in the body hereof and the somewhat more extensive listings in the bibliography should prove useful in guiding the reader to sources of information he may need but which does not appear herein.

One general remark should perhaps be appended. Throughout it is assumed that the material behaves in a linear elastic manner and that the resulting strains are sufficiently small that second-order effects can be neglected.

It is also assumed, except in cases specifically dealing with buckling, that deformations do not significantly alter the loading configuration. Briefly, except for cases of buckling, strictly linear behavior is postulated.

II. MEMBRANE THEORY

This analysis is thoroughly treated in the pressure vessel literature [4,10,17,19,25,26,27,32,33,34]. It is based upon considering a thin plate or sheet of revolution symmetrical about axis x-x (Fig. 1).

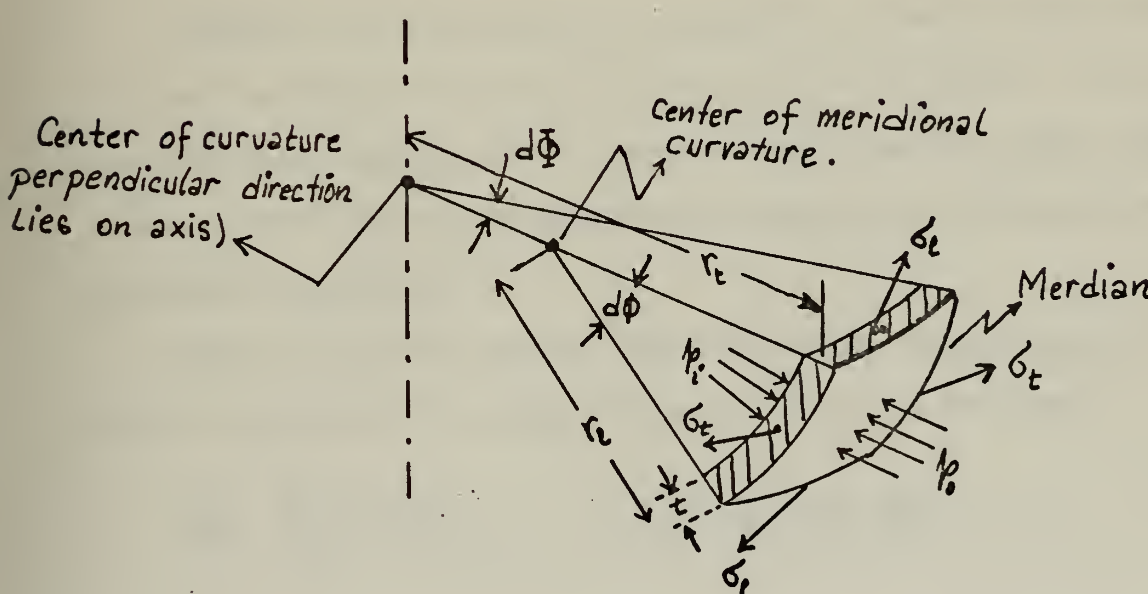


Fig. 1 Element of Axisymmetric membrane

If the wall thickness t of such a shell is small compared with its principal radii of curvature, the wall will have negligible bending resistance and acts as a membrane in which the stress is tangential to the middle surface of the wall

and uniformly distributed across its thickness. These stresses are called membrane stresses and can be found from equations of statics alone.

A basic equation describing the equilibrium of an element (see Fig. 1) in a direction normal to its surface is:

$$\frac{\sigma_\ell}{r_\ell} + \frac{\sigma_t}{r_t} = \frac{p_i - p_o}{t} \tag{1}$$

III. LONG THIN-WALLED CYLINDRICAL SHELLS UNDER INTERNAL AND EXTERNAL PRESSURE

Results for thin-walled cylindrical shells are reported by many authors [4,6,10,17,19,25,26,27,32,33,37]. It is assumed that the shell is long enough that the edge shear forces and bending moments have negligible influence on the section considered.

The well-known stress and deflection equations for thin-walled cylindrical shell can be written as follows:

$$\sigma_t = \frac{R_c}{t_c}(p_i - p_o) \qquad \sigma_t = \frac{R_c}{t_c}(p_i - p_o) \tag{1a,b}$$

$$\sigma_x = \frac{F}{2\pi R_c t_c} \qquad \sigma_x = \frac{1}{2}\cdot\frac{R_c}{t_c}(p_i - p_o) \tag{2a,b}$$

$$\delta_c = \frac{1}{E}\cdot\frac{(R_c)^2}{t_c}(p_i - p_o) - \frac{\nu F}{2\pi E t_c} \qquad \delta_c = \frac{2-\nu}{2E}\cdot\frac{(R_c)^2}{t_c}(p_i - p_o) \tag{3a,b}$$

Formulas (a) are for the case of a general axial tensile force F in the material of the cylinder. If this arises solely from pressure, then $F = (p_i - p_o)\pi R_c^2$ and formulas (b) result.

IV. LONG THIN-WALLED CYLINDRICAL SHELLS UNDER INTERNAL PRESSURE

The stress and deflection equations can be obtained from Equations 1,2, and 3 in Section III by putting $p_o = 0$. The simplified formulas are

$$\sigma_t = \frac{R_c}{t_c} p_i \qquad \sigma_t = \frac{R_c}{t_c} p_i \qquad (1a,b)$$

$$\sigma_x = \frac{F}{2\pi R_c t_c} \qquad \sigma_x = \frac{1}{2} \times \frac{R_c}{t_c} p_i \qquad (2a,b)$$

$$\delta_c = \frac{1}{E} \times \frac{(R_c)^2}{t_c} p_i - \frac{\nu F}{2\pi E t_c} \qquad \delta_c = \frac{2-\nu}{2E} \times \frac{(R_c)^2}{t_c} p_i \qquad (3a,b)$$

V. LONG THIN-WALLED CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE

Equations of stress and deflection can be obtained, as was done in Section IV, from equations 1, 2, and 3 in Section III by putting $p_i = 0$. The results are

$$\sigma_t = - \frac{R_c}{t_c} p_o \qquad \sigma_t = - \frac{R_c}{t_c} p_o \qquad (1a,b)$$

$$\sigma_x = \frac{F}{2\pi R_c t_c} \qquad \sigma_x = -\frac{1}{2} \cdot \frac{R_c}{t_c} p_o \tag{2a,b}$$

$$\delta_c = -\frac{1}{E} \cdot \frac{(R_c)^2}{t_c} p_o - \frac{\nu F}{2\pi E t_c} \qquad \delta_c = \frac{2-\nu}{2E} \cdot \frac{(R_c)^2}{t_c} p_o \tag{3a,b}$$

cf., also, Section XXXV.

VI. THIN-WALLED SPHERICAL SHELLS UNDER INTERNAL AND EXTERNAL PRESSURE

Stress and deflection equations for spherical shells can be found in many books and articles [4,6,10,17,19,25,27,33, 34,37]. Membrane equation 1 in Section II can be used to get the stress equation for thin-walled spherical shells by taking longitudinal and hoop radii to be equal and also taking longitudinal and hoop stress to be equal. Hence the equation can be written as follows;

$$\sigma = \frac{1}{2} \cdot \frac{R_s}{t_s} (p_i - p_o) \tag{1}$$

The deflection equation may be written as follows

$$\delta_s = \frac{1-\nu}{2E} \cdot \frac{(R_s)^2}{t_s} (p_i - p_o) \tag{2}$$

VII. THIN-WALLED SPHERICAL SHELLS UNDER INTERNAL PRESSURE

Stress and deflection equations can be obtained from equation 1 and 2 in Section VI by simply putting $p_o = 0$ as follows

$$\sigma = \frac{1}{2} \times \frac{R_s}{t_s} p_i. \qquad (1)$$

$$\delta_s = \frac{1-\nu}{2E} \cdot \frac{(R_s)^2}{t_s} p_i. \qquad (2)$$

VIII. THIN-WALLED SPHERICAL SHELLS UNDER EXTERNAL PRESSURE

Substituting $p_i = 0$ in equations (1) and (2) of Section VI gives

$$\sigma = -\frac{1}{2} \times \frac{R_s}{t_s} p_o \qquad (1)$$

$$\delta_s = -\frac{1-\nu}{2E} \cdot \frac{(R_s)^2}{t_s} p_o \qquad (2)$$

(Cf., also Section XLI.)

IX. LONG THICK-WALLED CYLINDRICAL PRESSURE VESSELS UNDER INTERNAL AND EXTERNAL PRESSURE

Thick-walled cylindrical vessels have been studied by many authors [4,6,19,33,34,36]. When the thickness of the cylindrical shell is relatively large, the stress variation

from inner to the outer surface becomes important. (The analysis in this section is known by the name of G. Lamé who first developed it.)

The membrane theory can not be applied. If a thick walled cylindrical vessel with constant thickness is subjected to uniform internal and external pressure, deformation will be symmetrical about the axis x-x. The axial stress σ_x is constant through the thickness; it must be determined by conditions of axial equilibrium. Stress, deformation, and deflection equations can be written as follows:

$$\sigma_r = \frac{(R_{ci})^2 p_i - (R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} - \frac{(R_{ci})^2 \, (R_{co})^2}{(R_{co})^2 - (R_{ci})^2} \, (p_i - p_o) \times \frac{1}{(r)^2} \tag{1}$$

$$\sigma_t = \frac{(R_{ci})^2 p_i - (R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} + \frac{(R_{ci})^2 \, (R_{co})^2}{(R_{co})^2 - (R_{ci})^2} \, (p_i - p_o) \times \frac{1}{(r)^2} \tag{2}$$

$$u = \frac{1-\nu}{E} \times \frac{(R_{ci})^2 p_i - (R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} \times r + \frac{1+\nu}{E} \times \frac{(R_{ci})^2 (R_{co})^2}{(R_{co})^2 - (R_{ci})^2} (p_i - p_o) \times \frac{1}{r} - \frac{\nu}{E} \sigma_x \, r \tag{3}$$

$$\delta_{ci} = u\Big|_{r=R_{ci}} = \frac{1-\nu}{E} \times \frac{(R_{ci})^2 p_i - (R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} \cdot R_{ci} + \frac{1+\nu}{E} \times \frac{(R_{ci})^2 (R_{co})^2}{(R_{co})^2 - (R_{ci})^2} (p_i - p_o) \cdot \frac{1}{R_{ci}} - \frac{\nu}{E} \sigma_x \, R_{ci} \tag{4}$$

$$\delta_{co} = u\Big)_{r=R_{co}} = \frac{1-\nu}{E} \times \frac{(R_{ci})^2 p_i - (R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} \times R_{co} + \frac{1+\nu}{E} \times \frac{(R_{ci})^2 (R_{co})^2}{(R_{co})^2 - (R_{ci})^2} (p_i - p_o) \times \frac{1}{R_{co}} - \frac{\nu}{E} \sigma_x R_{co} \tag{5}$$

Equations (4) and (5) give the outward displacement of inner surface and outer surface of thick-walled cylindrical vessel respectively.

X. LONG THICK-WALLED CYLINDRICAL PRESSURE VESSELS UNDER INTERNAL PRESSURE

Stress, deformation and deflection equations can be obtained from eqs. (1), (2), (3), (4) and (5) in Section IX by simply putting $p_o = 0$, so as to obtain the following results.

$$\sigma_r = \frac{(R_{ci})^2 p_i}{(R_{co})^2 - (R_{ci})^2} - \frac{(R_{ci})^2 (R_{co})^2 p_i}{(R_{co})^2 - (R_{ci})^2} \times \frac{1}{(r)^2} \tag{1}$$

$$\sigma_t = \frac{(R_{ci})^2 p_i}{(R_{co})^2 - (R_{ci})^2} + \frac{(R_{ci})^2 (R_{co})^2 p_i}{(R_{co})^2 - (R_{ci})^2} \times \frac{1}{(r)^2} \tag{2}$$

$$(\sigma_t)_{max} = \frac{(R_{ci})^2 + (R_{co})^2}{(R_{co})^2 - (R_{ci})^2} \times p_i \qquad \text{(at inside surface)} \tag{3}$$

$$u = \frac{1-\nu}{E} \times \frac{(R_{ci})^2 p_i}{(R_{co})^2-(R_{ci})^2} \times r + \frac{1+\nu}{E} \times \frac{(R_{ci})^2 (R_{co})^2 p_i}{(R_{co})^2-(R_{ci})^2} \times \frac{1}{r} - \frac{\nu}{E}\sigma_x r \qquad (4)$$

$$\delta_{ci} = u\Big)_{r=R_{ci}} = \frac{1}{E} \times R_{ci}\left[\frac{(R_{ci})^2+(R_{co})^2}{(R_{co})^2-(R_{ci})^2} + \nu\right] p_i - \frac{\nu}{E}\sigma_x R_{ci} \qquad (5)$$

$$\delta_{co} = u\Big)_{r=R_o} = \frac{2}{E} R_{ci}\left[\frac{R_{ci} R_{co}}{(R_{co})^2-(R_{ci})^2}\right] p_i - \frac{\nu}{E}\sigma_x R_{co} \qquad (6)$$

XI. LONG THICK-WALLED CYLINDRICAL PRESSURE VESSELS UNDER EXTERNAL PRESSURE

Substituting $p_i = 0$ in equations (1), (2), (3), (4), and (5) of Section IX we are led to the equations

$$\sigma_r = -\frac{(R_{co})^2 p_o}{(R_{co})^2-(R_{ci})^2} + \frac{(R_{ci})^2(R_{co})^2 p_o}{(R_{co})^2-(R_{ci})^2} \times \frac{1}{(r)^2} \qquad (1)$$

$$\sigma_t = -\frac{(R_{co})^2 p_o}{(R_{co})^2-(R_{ci})^2} - \frac{(R_{ci})^2(R_{co})^2}{(R_{co})^2-(R_{ci})^2} \times \frac{1}{(r)^2} \qquad (2)$$

$$(\sigma_t)_{max} = -2 \times \frac{(R_{co})^2}{(R_{co})^2-(R_{ci})^2} p_o \qquad \text{(at inside surface)} \quad (3)$$

$$u = -\frac{1-\nu}{E} \times \frac{(R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} \times r - \frac{1+\nu}{E} \times \frac{(R_{ci})^2 (R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} \times \frac{1}{r} - \frac{\nu}{E} \sigma_x r \tag{4}$$

$$\delta_{ci} = u\Big]_{r=R_{ci}} = -\frac{2}{E} R_{ci} \left[\frac{(R_{co})^2}{(R_{co})^2 - (R_{ci})^2} \right] p_o - \frac{\nu}{E} \sigma_x R_{ci} \tag{5}$$

$$\delta_{co} = u\Big]_{r=R_{co}} = -\frac{1}{E} R_o \left[\frac{(R_{ci})^2 + (R_{co})^2}{(R_{co})^2 - (R_{ci})^2} - \nu \right] p_o - \frac{\nu}{E} \sigma_x R_{co} \tag{6}$$

XII. THICK-WALLED SPHERICAL PRESSURE VESSELS UNDER INTERNAL AND EXTERNAL PRESSURE

If the thickness of a spherical vessel is relatively large, the stress variation from inner to outer surface can not be neglected. Therefore, membrane theory can not be applied. For this case the spherical vessel can be analyzed by applying equilibrium equations of statics to an element also considering relations of geometric compatibility [4,36]. Stresses, strains, etc., are functions only of radius and not of latitude or longitude on the sphere.

Equations for stresses and deflections can be written as follows:

$$\sigma_r = -\frac{(R_{si})^3 \left[(R_{so})^3 - (r)^3 \right]}{(r)^3 \left[(R_{so})^3 - (R_{si})^3 \right]} p_i - \frac{(R_{so})^3 \left[(r)^3 - (R_{si})^3 \right]}{(r)^3 \left[(R_{so})^3 - (R_{si})^3 \right]} p_o \tag{1}$$

$$\sigma_t = \sigma_l = \frac{1}{2} \times \frac{(R_{si})^3\left[2(r)^3 + (R_{so})^3\right]}{(r)^3\left[(R_{so})^3 - (R_{si})^3\right]} p_i - \frac{1}{2} \times \frac{(R_{so})^3\left[2(r)^3 + (R_{si})^3\right]}{(r)^3\left[(R_{so})^3 - (R_{si})^3\right]} p_o \tag{2}$$

$$u = \frac{r}{2E} \times \frac{2\nu(R_{si})^3\left[(R_{so})^3 - (r)^3\right] + (1-\nu)(R_{si})^3\left[2(r)^3 + (R_{si})^3\right]}{(r)^3\left[(R_{so})^3 - (R_{si})^3\right]} p_i + \frac{r}{2E} \times \frac{2\nu(R_{so})^3\left[(r)^3 - (R_{si})^3\right] - (1-\nu)(R_{so})^3\left[2(r)^3 + (R_{si})^3\right]}{(r)^3\left[(R_{so})^3 - (R_{si})^3\right]} p_o \tag{3}$$

$$\delta_{si} = u\Big|_{r=R_{si}} = \frac{1}{2E} \times R_{si}\left[2\nu + (1-\nu) \times \frac{2(R_{si})^3 + (R_{so})^3}{(R_{so})^3 - (R_{si})^3}\right] p_i - \frac{1}{2E} \times R_{si}\left[3(1-\nu) \times \frac{(R_{so})^3}{(R_{so})^3 - (R_{si})^3}\right] p_o \tag{4}$$

$$\delta_{so} = u\Big|_{r=R_{so}} = \frac{1}{2E} \times R_{so}\left[3(1-\nu) \times \frac{(R_{si})^3}{(R_{so})^3 - (R_{si})^3}\right] p_i + \frac{1}{2E} \times R_{so}\left[2\nu - (1-\nu) \times \frac{2(R_{so})^3 + (R_{si})^3}{(R_{so})^3 - (R_{si})^3}\right] p_o \tag{5}$$

XIII. THICK-WALLED SPHERICAL PRESSURE VESSELS UNDER INTERNAL PRESSURE

Stresses, deformation and deflection equations can be obtained from equations (1), (2), (3), (4), and (5) in Section XII by putting $p_o = 0$ as follows:

$$\sigma_r = -\frac{(R_{si})^3\left[(R_{so})^3-(r)^3\right]}{(r)^3\left[(R_{so})^3-(R_{si})^3\right]}\,p_i \tag{1}$$

$$\sigma_t = \sigma_\ell = \frac{1}{2}\times\frac{(R_{si})^3\left[2(r)^3+(R_{so})^3\right]}{(r)^3\left[(R_{so})^3-(R_{si})^3\right]}\,p_i \tag{2}$$

$$(\sigma_t)_{max} = \frac{1}{2}\times\frac{2(R_{si})^3+(R_{so})^3}{(R_{so})^3-(R_{si})^3}\,p_i \tag{3}$$

$$u = \frac{r}{2E}\times\frac{2\nu(R_{si})^3\left[(R_{so})^3-(r)^3\right]+(1-\nu)(R_{si})^3\left[2(r)^3+(R_{so})^3\right]}{(r)^3\left[(R_{so})^3-(R_{si})^3\right]}\,p_i \tag{4}$$

$$\delta_{si} = u\Big)_{r=R_{si}} = \frac{1}{2E}\cdot R_{si}\left[2\nu+(1-\nu)\times\frac{2(R_{si})^3+(R_{so})^3}{(R_{so})^3-(R_{si})^3}\right]p_i \tag{5}$$

$$\delta_{so} = u\Big]_{r=R_{so}} = \frac{1}{2E} \times R_{so} \left[3(1-\nu) \times \frac{(R_{si})^3}{(R_{so})^3 - (R_{si})^3} \right] p_i . \tag{6}$$

XIV. THICK-WALLED SPHERICAL PRESSURE VESSELS UNDER EXTERNAL PRESSURE

Substituting $p_i = 0$ in equations (1), (2), (3), (4), and (5) of Section XII gives

$$\sigma_r = - \frac{(R_{so})^3 \left[(r)^3 - (R_{si})^3 \right]}{(r)^3 \left[(R_{so})^3 - (R_{si})^3 \right]} p_o \tag{1}$$

$$\sigma_t = \sigma_\ell = -\frac{1}{2} \times \frac{(R_{so})^3 \left[2(r)^3 + (R_{si})^3 \right]}{(r)^3 \left[(R_{so})^3 - (R_{si})^3 \right]} p_o \tag{2}$$

$$(\sigma_t)_{max} = -\frac{1}{2} \times \frac{2(R_{so})^3 + (R_{si})^3}{(R_{so})^3 - (R_{si})^3} \times p_o \tag{3}$$

$$u = \frac{r}{2E} \times \frac{2\nu (R_{so})^3 \left[(r)^3 - (R_{si})^3 \right] - (1-\nu)(R_{so})^3 \left[2(r)^3 + (R_{si})^3 \right]}{(r)^3 \left[(R_{so})^3 - (R_{si})^3 \right]} p_o \tag{4}$$

$$\delta_{si} = u \Big)_{r=R_{si}} = -\frac{1}{2E} \times R_{si} \left[3(1-\nu) \times \frac{(R_{so})^3}{(R_{so})^3 - (R_{si})^3} \right] p_o \tag{5}$$

$$\delta_{so} = u \Big)_{r=R_{so}} = \frac{1}{2E} \times R_{so} \left[2\nu - (1-\nu) \times \frac{2(R_{so})^3 + (R_{si})^3}{(R_{so})^3 - (R_{si})^3} \right] p_o \tag{6}$$

XV. ANALYSIS OF THIN-WALLED CYLINDRICAL SHELLS WITHOUT CLOSURE UNDER END LOADING

This case is important in yielding coefficients (influence coefficients) required for dislocation analysis due to the effect of end closures or junctions with other shells, etc. The analysis of this case is given in Appendix 1 (page 115). Only certain results are presented here in the body of the thesis.

A. SEMI-INFINITE CASE

The end deflection $\delta_c^{\ 1}$ and rotation $\theta_c^{\ 1}$ can be written in terms of end shear H_1 and end moment M_1 (see Fig. 1) as follows

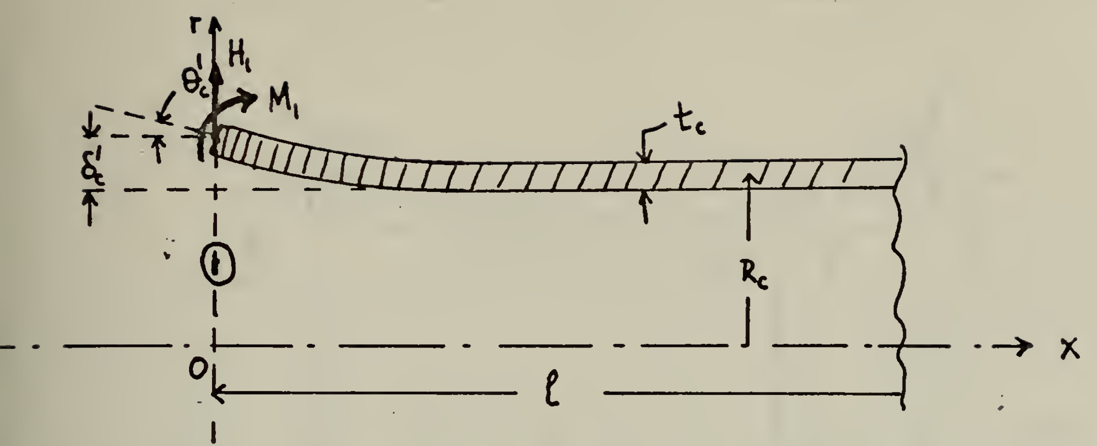


Fig. 1 - End region of semi-infinite cylinder

$$\delta_c^1 = \frac{1}{2(\beta_c)^3 D_c}\left[H_1 + \beta_c M_1\right] \tag{1}$$

$$\theta_c^1 = \frac{1}{2(\beta_c)^2 D_c}\left[H_1 + 2\beta_c M_1\right] \tag{2}$$

B. FINITE CASE

The deflections ($\delta_c^{\,1}, \delta_c^{\,2}$) and rotations ($\theta_c^{\,1}, \theta_c^{\,2}$) at ends 1 and 2 may be written in terms of the end shears (H_1, H_2) and moments (M_1, M_2) as follows (see Fig. 2).

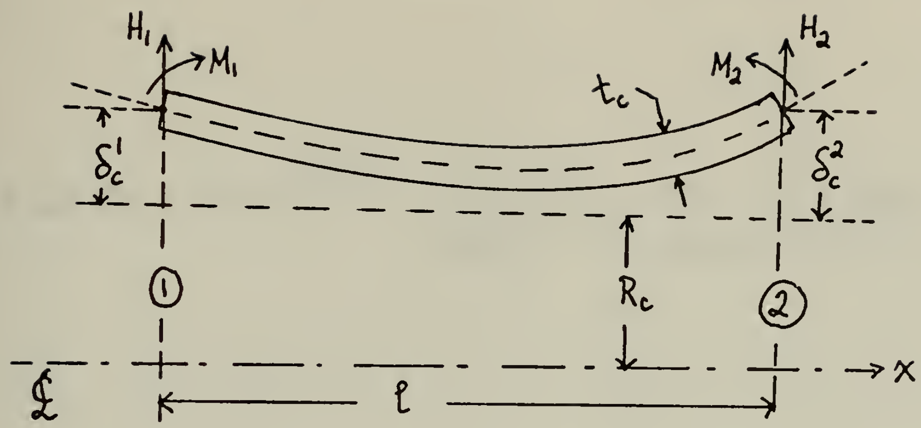


Fig. 2 - Cylinder of finite length

$$\delta_c^1 = \frac{\dfrac{H_1}{2(\beta_c)^3 D_c}\left[S^2 C^2 - s^2 c^2\right] + \dfrac{H_2}{2(\beta_c)^3 D_c}\left[S^2 c^2 - C^2 s^2\right] + \dfrac{M_1}{2(\beta_c)^2 D_c}\left[(C^2)^2 (s^2)^2 + (S^2)^2 (c^2)^2\right]}{(S^2)^2 - (s^2)^2} \qquad (1)$$

$$\theta_c^1 = \frac{\dfrac{H_1}{8(\beta_c)^3 D_c}\left[(S^2 c^2 - C^2 s^2)^2\right] + \dfrac{H_2}{4(\beta_c)^2 D_c}\left[S^2 s^2\right] + \dfrac{M_1}{4\beta_c D_c}\left[2 S^2 C^2 (c^2)^2 - s^2 c^2\right] + \beta_c \dfrac{M_2}{D_c}\left[C^2 s^2 - S^2 c^2\right]}{(S^2)^2 - (s^2)^2} \qquad (2)$$

$$\delta_c^2 = \delta_c^1(\mathcal{C}^2 c^2) + \frac{\theta_c^1}{2\beta_c}(\mathcal{C}^2 s^2 + \mathcal{S}^2 c^2) + \frac{H_1}{4(\beta_c)^3 D_c}(\mathcal{C}^2 s^2 - \mathcal{S}^2 c^2) + \frac{M_1}{2(\beta_c)^2 D_c}(\mathcal{S}^2 c^2) \tag{3}$$

$$\theta_c^2 = -\beta_c \delta_c^1(\mathcal{C}^2 s^2 - \mathcal{S}^2 c^2) + \theta_c^1(\mathcal{C}^2 c^2) + \frac{H_1}{2(\beta_c)^2 D_c}(\mathcal{S}^2 s^2) + \frac{M_1}{2\beta_c D_c}(\mathcal{C}^2 s^2 + \mathcal{S}^2 c^2) \tag{4}$$

In these equations the following abbreviations are used

$$\mathcal{S}^2 = \sinh \beta_c \ell$$

$$\mathcal{C}^2 = \cosh \beta_c \ell$$

$$s^2 = \sin \beta_c \ell$$

$$c^2 = \cos \beta_c \ell$$

The superscript 2 is not an exponent but, instead, indicates evaluation at the right end where $x = \ell$.

XVI. THIN-WALLED CYLINDRICAL SHELLS WITH PLATE-HEAD CLOSURE UNDER INTERNAL AND EXTERNAL PRESSURE

The analysis reported in this section and subsequent sections XVII through XXXI is for semi-infinite thin-wall

cylindrical shells which have various forms of closure at the accessible end. For a shell of finite length ℓ, if the product

$$\ell\beta_c = \ell \sqrt[4]{\frac{3(1-\nu^2)}{(R_c)^2(t_c)^2}}$$

is large (i.e., greater than 5.0, say) then the dislocation at one end has negligible effect on the conditions at the other end, and the analysis of a semi-infinite shell will provide satisfactory results. In shells which are not sufficiently long to justify this approximation, a more elaborate analysis is required. Brock [2] has treated the problem of cylinders composed of endwise juxtaposed short cylinders, each of constant thickness and properties, using matrix methods. The only type of closures explicitly considered in [2] are (a) hemispherical and (b) ideally rigid; however the procedure can easily accommodate to other types of closure.

Analysis of thin-walled semi-infinite cylindrical shells with flat head closure [7,15,23-a,28,37,42,43a], can be done by finding the deformations of each component separately, due to the pressure and arbitrary edge loads. The resultant deformations can be obtained by superposing the deformations due to the pressure and edge loads. Then the condition of continuity may be applied to provide enough equations to determine the edge loads necessary to produce continuous deformations (Fig. 1).

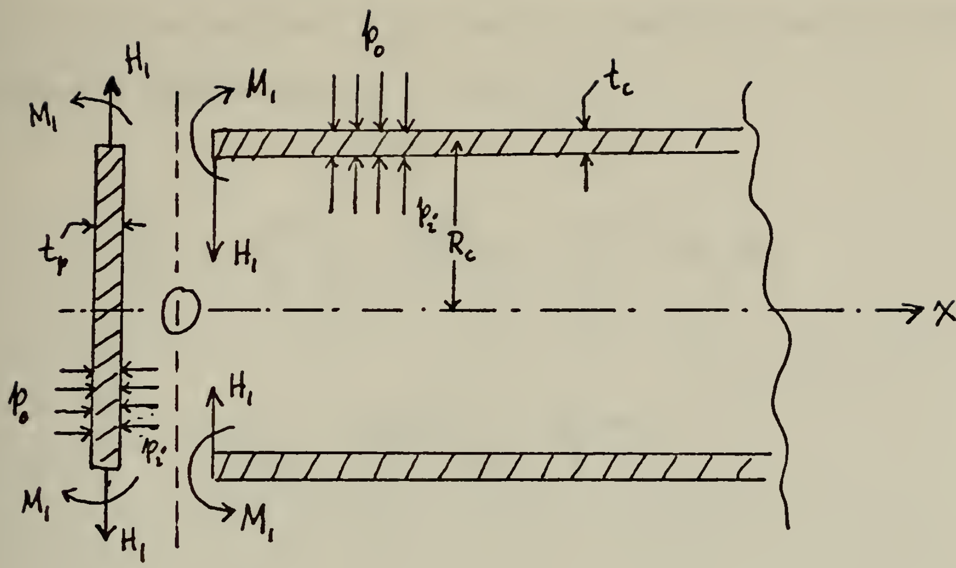


Fig. 1 Semi-infinite cylinder with flat plate closure

At the junction (1) edge deflection and rotation equations for the plate can be written as follows;

$$\delta_p^{1} = \frac{(1-\nu_p)}{E} \cdot \frac{R_c}{t_p} \cdot H_1 \tag{1}$$

$$\theta_p^{1} = -\frac{R_c}{D_p(1+\nu_p)} \cdot M_1 + \frac{1}{8} \cdot \frac{(R_c)^3}{D_p(1+\nu_p)}(p_i - p_o) \tag{2}$$

The edge deflections and rotations of the cylindrical shell can be written in terms of loadings and the edge influence coefficients as follows

$$\delta_c' = -\delta_c^{'H_1} H_1 + \delta_c^{'M_1} M_1 + \delta_c^{'P_1}(p_i - p_o) \tag{3}$$

$$\theta_c' = -\theta_c^{'H_1} H_1 + \theta_c^{'M_1} M_1 \tag{4}$$

The edge influence coefficients are:

$$\delta_c^{'H_1} = \frac{2\beta_c}{k_c} \qquad \theta_c^{'H_1} = \frac{2(\beta_c)^2}{k_c}$$

$$\delta_c^{'M_1} = \frac{2(\beta_c)^2}{k_c} \qquad \theta_c^{'M_1} = \frac{4(\beta_c)^3}{k_c} \tag{5}$$

$$\delta_c^{'P_1} = \frac{2-\nu_c}{2E} \times \frac{(R_c)^2}{t_c} \qquad \theta_c^{'P_1} = 0$$

Conditions of continuity give two simultaneous equations which permit one to determine the edge loads as follows;

$$-\left[\frac{(1-\nu_p)R_c}{Et_p}+\frac{2\beta_c(R_c)^2}{Et_c}\right]H_1+\frac{2(\beta_c)^2(R_c)^2}{Et_c}M_1+\frac{(2-\nu_c)(R_c)^2}{2Et_c}(p_i-p_o)=0 \tag{6}$$

$$-\frac{2(\beta_c)^2(R_c)^2}{Et_c}H_1+\left[\frac{4(\beta_c)^3(R_c)^2}{Et_c}+\frac{R_c}{D_p(1+\nu_p)}\right]M_1-\frac{(R_c)^3}{8D_p(1+\nu_p)}(p_i-p_o)=0 \tag{7}$$

Once the edge loads are known, the stresses can be obtained easily by adding the contributions from the pressure and edge bending moment.

XVII. THIN-WALLED CYLINDRICAL SHELLS WITH PLATE-HEAD CLOSURE UNDER INTERNAL PRESSURE

Substituting $p_o = 0$ into the equations (1), (2), (3), (4), (6), and (7) respectively of Section XVI gives;

$$\delta_p' = \frac{(1-\nu_p)R_c}{Et_p}H_1 \tag{1}$$

$$\theta_p' = \frac{R_c}{D_p(1+\nu_p)}M_1-\frac{(R_c)^3}{8D_p(1+\nu_p)}p_i \tag{2}$$

$$\delta_c' = -\delta_c^{\prime H_1} H_1 + \delta_c^{\prime M_1} M_1 + \delta_c^{\prime P_1} p_{2}. \quad (3)$$

$$\theta_c' = -\theta_c^{\prime H_1} H_1 + \theta_c^{\prime M_1} M_1 \quad (4)$$

$$-\left[\frac{(1-\nu_p) R_c}{E t_p} + \frac{2\beta_c (R_c)^2}{E t_c}\right] H_1 + \frac{2(\beta_c)^2 (R_c)^2}{E t_c} M_1 + \frac{(2-\nu_c)}{2 E t_c} p_{2} = 0 \quad (5)$$

$$-\frac{2(\beta_c)^2 (R_c)^2}{E t_c} H_1 + \left[\frac{4(\beta_c)^3 (R_c)^2}{E t_c} + \frac{R_c}{D_p (1+\nu_p)}\right] M_1 - \frac{(R_c)^3}{8 D_p (1+\nu_p)} p_{i} = 0 \quad (6)$$

XVIII. THIN-WALLED CYLINDRICAL SHELLS WITH FLAT-HEAD CLOSURE UNDER EXTERNAL PRESSURE

Substituting $p_i = 0$ into the equations (1), (2), (3), (4), (6) and (7) in Section XVI respectively

$$\delta_p' = \frac{(1-\nu_p) R_c}{E t_p} H_1 \quad (1)$$

$$\theta_p' = \frac{R_c}{D_p(1+\nu_p)} M_1 + \frac{(R_c)^3}{8D_p(1+\nu_p)} p_o \tag{2}$$

$$\delta_c' = -\delta_c^{'H_1} H_1 + \delta_c^{'M_1} M_1 - \delta_c^{'P_1} p_o \tag{3}$$

$$\theta_c' = -\theta_c^{'H_1} H_1 + \theta_c^{'M_1} M_1 \tag{4}$$

$$-\left[\frac{(1-\nu_p)R_c}{Et_p} + \frac{2\beta_c(R_c)^2}{Et_c}\right]H_1 + \frac{2(\beta_c)^2(R_c)^2}{Et_c}M_1 - \frac{(2-\nu_c)}{2Et_c} p_o = 0 \tag{5}$$

$$-\frac{2(\beta_c)^2(R_c)^2}{Et_c}H_1 + \left[\frac{4(\beta_c)^3(R_c)^2}{Et_c} + \frac{R_c}{D_p(1+\nu_p)}\right]M_1 + \frac{(R_c)^3}{8D_p(1+\nu_p)} p_o = 0 \tag{6}$$

XIX. THIN-WALLED CYLINDRICAL SHELL WITH CONICAL-HEAD CLOSURE UNDER INTERNAL AND EXTERNAL PRESSURE

The edge loads H_1 and M_1 can be determined from compatibility of displacement and rotation at the junction (Fig. 1). When the edge loads are known, the shear, circumferential and axial stresses can be determined at the junction [37,42,42a, 43]. Each stress is divided by $\frac{(p_i-p_o)d_c}{2t_c}$ (the circumferential

stress in a thin unrestrained cylinder under uniform pressure) to form a stress index, denoted by I. The stress index has two advantages. It reduces the magnitude of the results and enables one to tell immediately whether a stress is greater or less than the hoop stress in a thin cylinder.

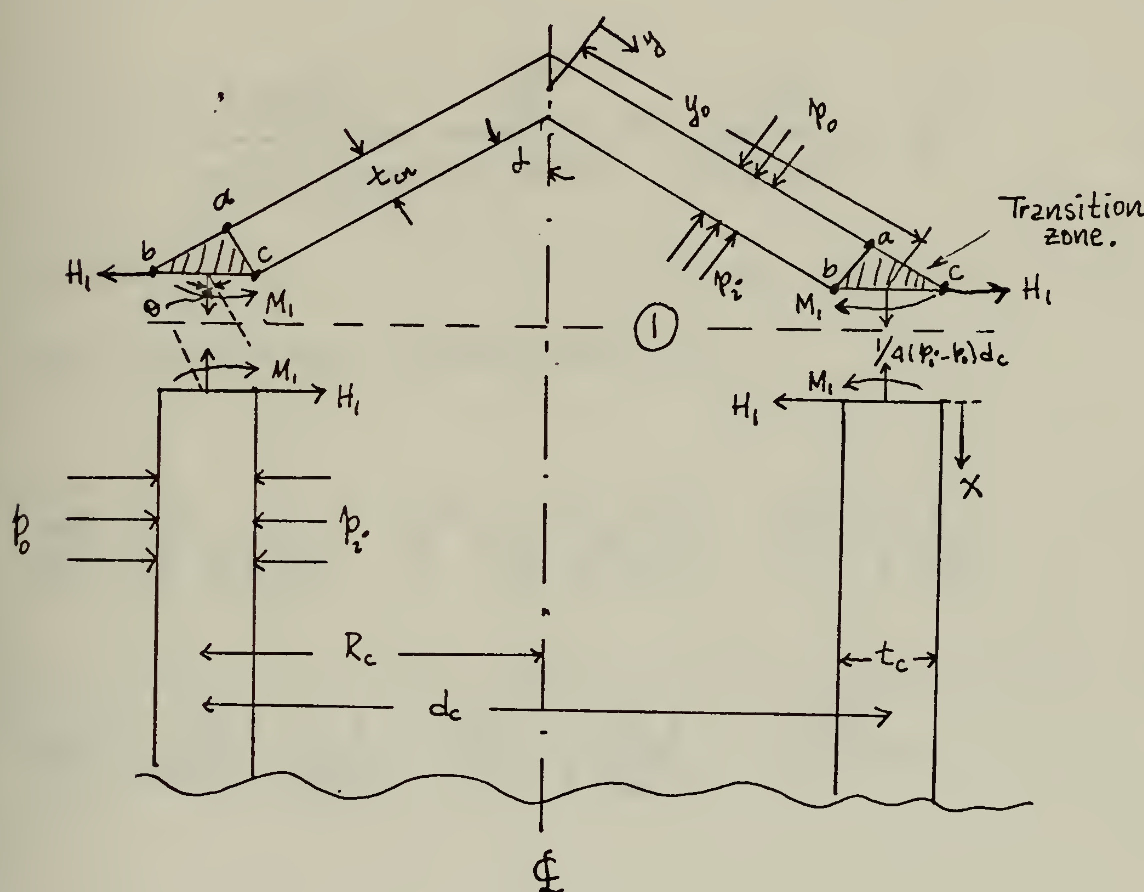


Fig. 1 Semi-infinite cylinder with conical closure

Some discrepency is to be expected between the theoretical results and the experimental results because of the effect of a transition zone a,b,c, (Fig. 1) where conical shell thickness may be considered as variable.

The edge deflection and rotation equations for the cylindrical shell can be written in terms of edge influence coefficients as follows:

$$\frac{E t_c \delta_c'}{(p_i - p_o)(d_c)^2} = \frac{\delta_c^{'M_1}}{(p_i - p_o)(d_c)^2} M_1 + \frac{\delta_c^{'H_1}}{(p_i - p_o) d_c} H_1 + \delta_c^{'P_1} \tag{1}$$

$$\frac{E (t_c)^2 \theta_c'}{(m)^2 (p_i - p_o)(d_c)^2} = \frac{\theta_c^{'M_1}}{(p_i - p_o)(d_c)^2} M_1 + \frac{\theta_c^{'H_1}}{(p_i - p_o) d_c} H_1 + \theta_c^{'P_1} \tag{2}$$

The edge influence coefficients are:

$$\begin{aligned}
&\delta_c^{'M_1} = -\frac{(m)^2}{2} \times \frac{d_c}{t_c} \times \frac{t_{cn}}{t_c} \qquad \delta_c^{'H_1} = -\frac{m}{2} \times \frac{t_{cn}}{t_c} \sqrt{\frac{d_c}{t_c}} \qquad \delta_c^{'P_1} = \frac{2 - \nu}{8} \times \frac{t_{cn}}{t_c} \\
&\theta_c^{'M_1} = -m \sqrt{\frac{d_c}{t_c}} \times \frac{(t_{cn})^2}{(t_c)^2} \qquad \theta_c^{'H_1} = -\frac{1}{2} \times \frac{(t_{cn})^2}{(t_c)^2} \qquad \theta_c^{'P_1} = 0
\end{aligned} \tag{3}$$

(note that $m = \sqrt[4]{12(1 - \nu^2)}$)

Similarly the edge deflection and rotation equations for the conical shell can be written in terms of edge influence as follows:

$$\frac{E t_{cn}\,\delta_{cn}^{\prime}}{(p_i^{\cdot}-p_o)(d_c)^2} = \frac{\delta_{cn}^{\prime M_1}}{(p_i^{\cdot}-p_o)(d_c)^2} M_1 + \frac{\delta_{cn}^{\prime H_1}}{(p_i^{\cdot}-p_o)d_c} H_1 + \delta_{cn}^{\prime P_1} \tag{4}$$

$$\frac{E(t_{cn})^2\,\theta_{cn}^{\prime}}{(m)^2(p_i^{\cdot}-p_o)(d_c)^2} = \frac{\theta_{cn}^{\prime M_1}}{(p_i^{\cdot}-p_o)(d_c)^2} M_1 + \frac{\theta_{cn}^{\prime H_1}}{(p_i^{\cdot}-p_o)d_c} H_1 + \theta_{cn}^{\prime P_1} \tag{5}$$

The edge influence coefficients are:

$$\delta_{cn}^{\prime M_1} = \frac{\theta_{cn}^{\prime H_1}(\xi_o)^2\tan\alpha\,\sin\alpha}{2} \qquad \delta_{cn}^{\prime H_1} = \frac{\frac{(\xi_o)^2 B}{4} - (\nu)^2 G}{C + 2\nu G}\,\sin\alpha \qquad \delta_{cn}^{\prime P_1} = \frac{2-\nu}{8}\sec\alpha - \frac{\delta_{cn}^{\prime H_1}}{4}\tan\alpha + \frac{3\theta_{cn}^{\prime H_1}(1+\nu)\sec\alpha}{(\xi_o)^2}$$

(6)

$$\theta_{cn}^{\prime M_1} = \frac{2G}{C+2\nu G}\cdot\frac{(\xi_o)^2\tan\alpha}{4} \qquad \theta_{cn}^{\prime H_1} = \frac{\frac{A}{2}}{C+2\nu G} \qquad \theta_{cn}^{\prime P_1} = \frac{6(1+\nu)}{(\xi_o)^2}\theta_{cn}^{\prime M_1}\operatorname{cosec}^2\alpha - \theta_{cn}^{\prime H_1}\times\frac{\tan\alpha}{4} - \frac{3\sec\alpha\operatorname{cosec}\alpha}{2(\xi_o)^2}$$

where

ξ_o = Cone parameter at the base = $m\sqrt{2\left(\frac{d_c}{t_{cn}}\right)\cot\alpha\operatorname{cosec}\alpha}$

$$A = \xi_o(\mathrm{ber}_2'\xi_o\,\mathrm{bei}_2\xi_o - \mathrm{bei}_2'\xi_o\,\mathrm{ber}_2\xi_o)$$

$$B = (\mathrm{ber}_2' \xi_0)^2 + (\mathrm{bei}_2' \xi_0)^2$$

$$C = \xi_0 (\mathrm{ber}_2 \xi_0 \, \mathrm{ber}_2' \xi_0 + \mathrm{bei}_2 \xi_0 \, \mathrm{bei}_2' \xi_0)$$

$$G = (\mathrm{ber}_2 \xi_0)^2 + (\mathrm{bei}_2 \xi_0)^2$$

The formulas for the Bessel (type) functions ber and bei can be found from [24-a].

It is not necessary to calculate the edge influence coefficients for the conical shell; numerical values have been tabulated in [43], including proper algebraic signs so that the following formulas are correct

Conditions of continuity give the following equations

$$\frac{\delta_c^{\,1M_1} - \delta_{cn}^{\,1M_1}}{(p_i - p_o)(d_c)^2} M_1 + \frac{\delta_c^{\,1H_1} - \delta_{cn}^{\,1H_1}}{(p_i - p_o) d_c} H_1 + (\delta_c^{\,1P_1} - \delta_{cn}^{\,1P_1}) = 0 \tag{7}$$

$$\frac{\theta_c^{\,1M_1} - \theta_{cn}^{\,1M_1}}{(p_i - p_o)(d_c)^2} M_1 + \frac{\theta_c^{\,1H_1} - \theta_{cn}^{\,1H_1}}{(p_i - p_o) d_c} H_1 - \theta_{cn}^{\,1P_1} = 0 \tag{8}$$

Simultaneous solution of equations (7) and (8) gives the edge loads.

Stress index equations for the cylindrical shell can be written as follows;

$$I_c^{(\tau)_1} = 3\left|\frac{H_1}{(p_i - p_o)d_c}\right| \tag{9}$$

$$I_c^{(\sigma_x)_1} = \frac{1}{2} + 12 \times \frac{d_c}{t_c}\left|\frac{M_1}{(p_i - p_o)(d_c)^2}\right| \tag{10}$$

$$I_c^{(\sigma_t)_1} = \left|1 - 2m\sqrt{\frac{d_c}{t_c}} \times \frac{H_1}{(p_i - p_o)d_c}(1+\mu)\right| + 12\nu \times \frac{d_c}{t_c}\left|\frac{M_1}{(p_i - p_o)(d_c)^2}\right| \tag{11}$$

Similarly stress index equations for the conical shell can be written as follows:

$$I_{cn}^{(\tau)_1} = 3\,\frac{t_c}{t_{cn}}\left|\frac{H_1}{(p_i - p_o)d_c}\cos\alpha - \frac{1}{4}\sin\alpha\right| \tag{12}$$

$$I_{cn}^{(\sigma_x)_1} = 2 \times \frac{t_c}{t_{cn}}\left|\frac{1}{4}\cos\alpha + \frac{H_1}{(p_i - p_o)d_c}\sin\alpha\right| + 12 \times \frac{d_c}{t_{cn}} \times \frac{t_c}{t_{cn}}\left|\frac{M_1}{(p_i - p_o)(d_c)^2}\right| \tag{13}$$

$$I_{cn}^{(\sigma_t)_1} = 2 \cdot \frac{t_c}{t_{cn}} \left\{ (m)^2 \sin\alpha \left| \frac{M_1}{(p_i - p_o)(d_c)^2} \left[\theta_{cn}^{|M_1} + \frac{3\nu(\xi_o)^2 \tan\alpha}{(m)^4} \right] + \frac{}{(p_i - p_o) d_c} H_1 + \theta_{cn}^{|P_1} \right| \right. \tag{14}$$

$$\left. + \left| \frac{2\,\delta_{cn}^{|M_1}}{(p_i - p_o)(d_c)^2} M_1 + \frac{2\delta_{cn}^{|H_1} + \nu \sin\alpha}{(p_i - p_o) d_c} H_1 + 2\,\delta_{cn}^{|P_1} + \frac{\nu \cos\alpha}{4} \right| \right\}$$

These stress indices are tabulated in [43].

XX. THIN-WALLED CYLINDRICAL SHELLS WITH CONICAL CLOSURE UNDER INTERNAL PRESSURE

Substituting $p_o = 0$ into the equations (7), (8), (9), (10), (11), (12), (13) and (14) of Section XIX gives

$$\frac{\delta_c^{|M_1} - \delta_{cn}^{|M_1}}{p_i \cdot (d_c)^2} M_1 + \frac{\delta_c^{|H_1} - \delta_{cn}^{|H_1}}{p_i \cdot d_c} H_1 + \left(\delta_c^{|P_1} - \delta_{cn}^{|P_1} \right) = 0 \tag{1}$$

$$\frac{\theta_c^{|M_1} - \theta_{cn}^{|M_1}}{p_i \cdot (d_c)^2} M_1 + \frac{\theta_c^{|H_1} - \theta_{cn}^{|H_1}}{p_i \cdot d_c} H_1 - \theta_{cn}^{|P_1} = 0 \tag{2}$$

$$I_c^{(\tau)_1} = 3 \left| \frac{H_1}{p_i \cdot d_c} \right| \tag{3}$$

$$I_c^{(\sigma_x)_1} = \frac{1}{2} + 12 \cdot \frac{d_c}{t_c} \left| \frac{M_1}{p_i \cdot (d_c)^2} \right| \tag{4}$$

$$I_c^{(\sigma_t)_1} = \left| 1 - 2m\sqrt{\frac{d_c}{t_c}} \cdot \frac{H_1}{p_i \cdot d_c}(1+\mu) \right| + 12\nu \cdot \frac{d_c}{t_c} \left| \frac{M_1}{p_i \cdot (d_c)^2} \right| \tag{5}$$

$$I_{cn}^{(\tau)_1} = 3\, \frac{t_c}{t_{cn}} \left| \frac{H_1}{p_i \cdot d_c} \cos\alpha - \frac{1}{4}\sin\alpha \right| \tag{6}$$

$$I_{cn}^{(\sigma_x)_1} = 2 \cdot \frac{t_c}{t_{cn}} \left| \frac{1}{4}\cos\alpha + \frac{H_1}{p_i \cdot d_c}\sin\alpha \right| + 12 \cdot \frac{d_c}{t_{cn}} \cdot \frac{t_c}{t_{cn}} \left| \frac{M_1}{p_i \cdot (d_c)^2} \right| \tag{7}$$

$$I_{cn}^{(\sigma_t)_1} = 2 \cdot \frac{t_c}{t_{cn}} \left\{ (m)^2 \sin\alpha \left| \frac{M_1}{p_i \cdot (d_c)^2} \left[\theta_{cn}^{1M_1} + \frac{3\nu(\xi_0)^2 \tan\alpha}{(m)^4} \right] + \frac{\theta_{cn}^{1H_1}}{p_i \cdot d_c} H_1 + \theta_{cn}^{1P_1} \right| + \left| \frac{2\delta_{cn}^{1M_1}}{p_i \cdot (d_c)^2} M_1 + \frac{2\delta_{cn}^{1H_1} + \nu\sin\alpha}{p_i \cdot d_c} H_1 + 2\delta_{cn}^{1P_1} + \frac{\nu\cos\alpha}{4} \right| \right\} \tag{8}$$

XXI. THIN-WALLED CYLINDRICAL SHELLS WITH CONICAL-HEAD CLOSURE UNDER EXTERNAL PRESSURE

Substituting $p_i = 0$ into the equations (7), (8), (9), (10), (11), (12), (13), and (14) of Section XIX gives

$$\frac{\delta_c^{|M_1} - \delta_{cn}^{|M_1}}{p_o (d_c)^2} M_1 + \frac{\delta_c^{|H_1} - \delta_{cn}^{|H_1}}{p_o d_c} H_1 + \left(\delta_c^{|P_1} - \delta_{cn}^{|P_1} \right) = 0 \qquad (1)$$

$$\frac{\theta_c^{|M_1} - \theta_{cn}^{|M_1}}{p_o (d_c)^2} M_1 + \frac{\theta_c^{|H_1} - \theta_{cn}^{|H_1}}{p_o d_c} H_1 - \theta_{cn}^{|P_1} = 0 \qquad (2)$$

$$I_c^{(\tau)_1} = 3 \left| \frac{H_1}{(-p_o) d_c} \right| \qquad (3)$$

$$I_c^{(\sigma_x)_1} = \frac{1}{2} + 12 \times \frac{d_c}{t_c} \left| \frac{M_1}{(-p_o)(d_c)^2} \right| \qquad (4)$$

$$I_c^{(\sigma_t)_1} = \left| 1 - 2m\sqrt{\frac{d_c}{t_c}} \cdot \frac{H_1}{(-p_o) d_c} (1+\mu) \right| + 12\nu \times \frac{d_c}{t_c} \left| \frac{M_1}{(-p_o)(d_c)^2} \right| \qquad (5)$$

$$I_{cn}^{(\tau)_1} = 3 \frac{t_c}{t_{cn}} \left| \frac{H_1}{(-p_o) d_c} \cos\alpha - \frac{1}{4} \sin\alpha \right| \qquad (6)$$

$$I_{cn}^{(\sigma_x)_1} = 2\cdot\frac{t_c}{t_{cn}}\left|\frac{1}{4}\cos\alpha + \frac{H_1}{(-P_0)d_c}\sin\alpha\right| + 12\cdot\frac{d_c}{t_{cn}}\cdot\frac{t_c}{t_{cn}}\left|\frac{M_1}{(-P_0)(d_c)^2}\right| \tag{7}$$

$$I_{cn}^{(\sigma_t)_1} = 2\cdot\frac{t_c}{t_{cn}}\left\{(m)^2\sin\alpha\left|\frac{M_1}{(-P_0)(d_c)^2}\left[\theta_{cn}^{1M_1} + \frac{3\nu(\xi_0)^2\tan\alpha}{(m)^4}\right] + \frac{\theta_{cn}^{1H_1}}{(-P_0)d_c}H_1 + \theta_{cn}^{1P_1}\right|\right. \tag{8}$$

$$\left. + \left|\frac{2\delta_{cn}^{1M_1}}{(-P_0)(d_c)^2}M_1 + \frac{2\delta_{cn}^{1H_1} + \nu\sin\alpha}{(-P_0)d_c}H_1 + 2\delta_{cn}^{1P_1} + \frac{\nu\cos\alpha}{4}\right|\right\}$$

XXII. TWO ADJACENT CYLINDRICAL OR HEMISPHERICAL SHELLS OR CYLINDRICAL SHELL WITH HEMISPHERICAL HEAD CLOSURE UNDER INTERNAL AND EXTERNAL PRESSURE

The case of two abutting cylinders is contained in what follows by taking the parameters $n_1 = n_2 = 2$. The case of abutting hemispherical shells is given by $n_1 = n_2 = 1$. The case of a cylindrical shell closed by a hemispherical shell is given by taking $n_1 = 2n_2 = 2$ or $n_2 = 2n_1 = 2$. That all these cases can be treated by the same method, i.e., by assuming, in effect, that the sphere behaves near its edge as if it were a cylinder of equal radius, results from the fact that the edge dislocation effects are confined to a narrow band near the edge, and in this band the spherical shell is actually very nearly a cylinder. The difference between the several cases thus arises solely in the difference between the change in radius due to pressurization in a cylinder and a sphere [4,15,20,25,33,37,42].

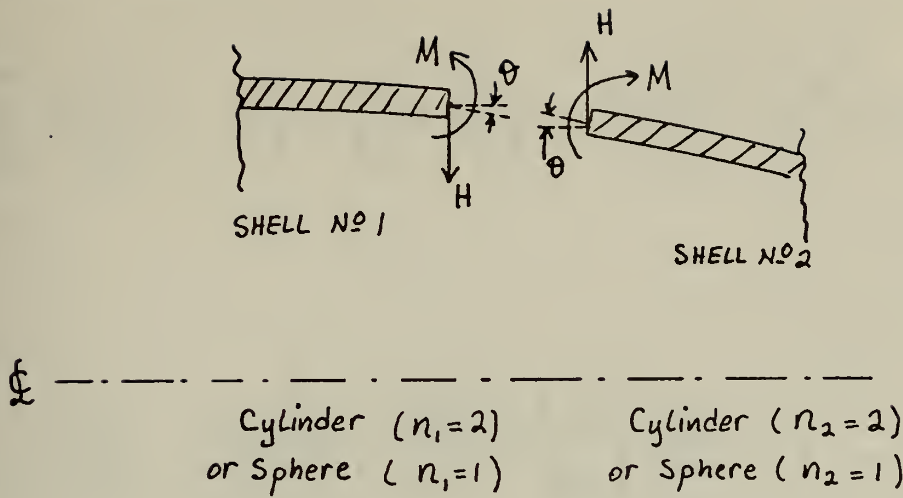


Fig. 1 Abutting cylindrical or hemispherical shell elements

We have

$$\delta = \frac{2(\beta_1)^2}{k_1} M - \frac{2\beta_1}{k_1} H + \left(\frac{p_i - p_o}{2k_1}\right)(n_1 - \nu_1) = \frac{2(\beta_2)^2}{k_2} M + \frac{2\beta_2}{k_2} H + \left(\frac{p_i - p_o}{2k_2}\right)(n_2 - \nu_2) \tag{1}$$

$$\theta = \frac{2(\beta_1)^2}{k_1} H - \frac{4(\beta_1)^3}{k_1} M = \frac{2(\beta_2)^2}{k_2} H + \frac{4(\beta_2)^3}{k_2} M \tag{2}$$

and thus obtain

$$\left[\frac{(\beta_1)^2}{k_1} - \frac{(\beta_2)^2}{k_2}\right] M - \left(\frac{\beta_1}{k_1} + \frac{\beta_2}{k_2}\right) H + \left(\frac{n_1 - \nu_1}{k_1} - \frac{n_2 - \nu_2}{k_2}\right)\left(\frac{p_i - p_0}{4}\right) = 0 \tag{3}$$

$$\left[\frac{(\beta_1)^3}{k_1} + \frac{(\beta_2)^3}{k_2}\right] M - \left[\frac{(\beta_1)^2}{k_1} - \frac{(\beta_2)^2}{k_2}\right]\left(\frac{H}{2}\right) = 0 \tag{4}$$

which can be solved to yield M and H.

If the thickness and the material of each component is the same, then obviously

$$M = 0 \tag{5}$$

$$H = (n_1 - n_2)\left(\frac{p_i - p_0}{8\beta}\right) \tag{6}$$

XXIII. THIN-WALLED CYLINDRICAL SHELLS WITH HEMISPHERICAL-HEAD CLOSURE, WITH SAME MATERIAL AND SAME THICKNESS UNDER INTERNAL PRESSURE

The formulas of Section XXII give

$$M = 0 \tag{1}$$

$$H = \frac{p_i}{8\beta} \tag{2}$$

XXIV. THIN-WALLED CYLINDRICAL SHELLS WITH HEMISPHERICAL-HEAD CLOSURE, WITH SAME MATERIAL AND SAME THICKNESS UNDER EXTERNAL PRESSURE

The formulas of Section XXII give

$$M = 0 \tag{1}$$

$$H = -\frac{p_o}{8\beta} \tag{2}$$

XXV. THIN-WALLED CYLINDRICAL SHELLS WITH SPHERICAL HEAD CLOSURE UNDER INTERNAL AND EXTERNAL PRESSURE

Since the loading is axisymmetrical, the deflection and rotation equations of both cylindrical and spherical shells at the junction can be written in terms of edge influence

coefficients [13,18,37,42]. Then the conditions of continuity for the junction (Fig. 1) give simultaneous equations for edge shear force and bending moment. If the angle $\phi = 90°$ so that the spherically curved closure becomes a hemisphere, the formulas herein simplify greatly and correspond to those already given in Section XXII.

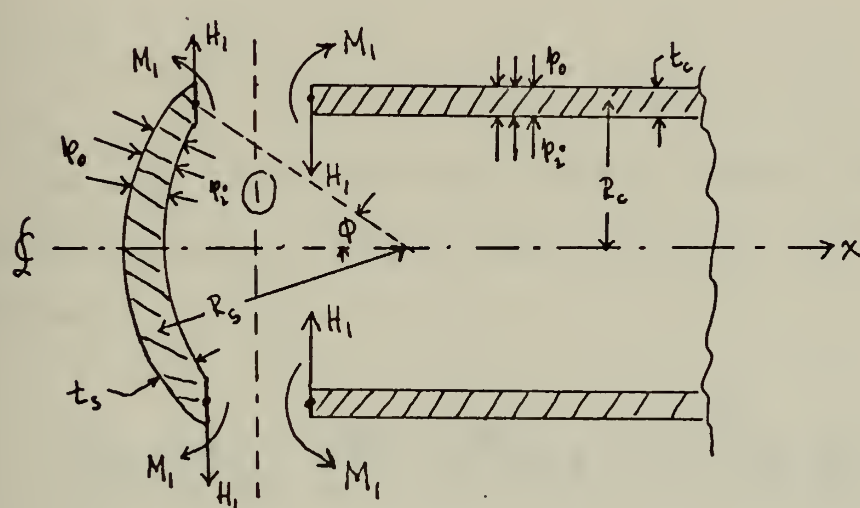


Fig. 1 Semi-infinite cylinder with spherically shaped end closure

The edge deflection and rotation for the cylindrical shell can be written in terms of edge influence coefficients as follows

$$\delta_c^1 = \delta_c^{1M_1} M_1 + \delta_c^{1H_1} H_1 + \delta_c^{1P_1}(p_i - p_o) \tag{1}$$

$$\theta_c^1 = \theta_c^{1M_1} M_1 + \theta_c^{1H_1} H_1 \tag{2}$$

The edge influence coefficients are:

$$\delta_c^{1M_1} = \frac{1}{2(\beta_c)^2 D_c} \qquad \delta_c^{1H_1} = \frac{1}{2(\beta_c)^3 D_c} \qquad \delta_c^{1P_1} = \frac{(2-\nu_c)(R_c)^2}{2Et_s}$$

$$\theta_c^{1M_1} = \frac{1}{\beta_c D_c} \qquad \theta_c^{1H_1} = \frac{1}{2(\beta_c)^2 D_c} \qquad \theta_c^{1P_1} = 0 \tag{3}$$

The edge deflection and rotation for the spherical shell can be written in terms of edge influence coefficients as follows:

$$\delta_s^1 = \delta_s^{1M_1} M_1 + \delta_s^{1H_1} H_1 + \delta_s^{1P_1}(p_i - p_o) + \delta_s^{1F_1}(p_i - p_o) \tag{4}$$

$$\theta_s^1 = \theta_s^{1M_1} M_1 + \theta_s^{1H_1} H_1 + \theta_s^{1F_1}(p_i - p_o) \tag{5}$$

The edge influence coefficients are:

$$\delta_s^{1M_1} = \frac{2(\lambda_s)^2 \sin\phi}{Et_s} \qquad \delta_s^{1H_1} = \frac{2\lambda_s R_s (\sin\phi)^2}{Et_s}$$

$$\delta_s^{1P_1} = \frac{(1-\nu_s)(R_s)^2 \sin\phi}{2Et_s} \qquad \delta_s^{1F_1} = \frac{\lambda_s (R_s)^2 (\sin\phi)^2 \cos\phi}{Et_s}$$

$$\theta_s^{1M_1} = -\frac{4(\lambda_s)^3}{ER_s t_s} \qquad \theta_s^{1H_1} = \frac{2(\lambda_s)^2 \sin\phi}{Et_s} \tag{6}$$

$$\theta_s^{1P_1} = 0 \qquad \theta_s^{1F_1} = \frac{(\lambda_s)^2 R_s \sin\phi \cos\phi}{Et_s}$$

Conditions of continuity give the following simultaneous equations

$$(\delta_c^{1M_1} - \delta_s^{1M_1})M_1 + (\delta_c^{1H_1} - \delta_s^{1H_1})H_1 = (\delta_s^{1P_1} - \delta_c^{1P_1})(p_1^{\circ} - p_0) + \delta_s^{1F_1}(p_2^{\circ} - p_0) \tag{7}$$

$$(\theta_c^{1M_1} - \theta_s^{1M_1})M_1 + (\theta_c^{1H_1} - \theta_s^{1H_1})H_1 = \theta_s^{1F_1}(p_2 - p_0) \tag{8}$$

XXVI. THIN-WALLED CYLINDRICAL SHELLS WITH SPHERICAL-HEAD CLOSURE UNDER INTERNAL PRESSURE

Substituting $p_o = 0$ into the equations (1), (2), (4), (5), (7), and (8) in Section XXV give:

$$\delta_c' = \delta_c'^{M_1} M_1 + \delta_c'^{H_1} H_1 + \delta_c'^{P_1} p_2. \tag{1}$$

$$\theta_c' = \theta_c'^{M_1} M_1 + \theta_c'^{H_1} H_1 \tag{2}$$

$$\delta_s' = \delta_s'^{M_1} M_1 + \delta_s'^{H_1} H_1 + \delta_s'^{P_1} p_2 + \delta_s'^{F_1} p_2. \tag{3}$$

$$\theta_s' = \theta_s'^{M_1} M_1 + \theta_s'^{H_1} H_1 + \theta_s'^{F_1} p_2. \tag{4}$$

$$(\delta_c'^{M_1} - \delta_s'^{M_1}) M_1 + (\delta_c'^{H_1} - \delta_s'^{H_1}) H_1 = (\delta_s'^{P_1} - \delta_c'^{P_1}) p_2 + \delta_c'^{F_1} p_2. \tag{5}$$

$$(\theta_c'^{M_1} - \theta_s'^{M_1}) M_1 + (\theta_c'^{H_1} - \theta_s'^{H_1}) H_1 = \theta_s'^{F_1} p_2. \tag{6}$$

XXVII. THIN-WALLED CYLINDRICAL SHELLS WITH SPHERICAL-HEAD CLOSURE UNDER EXTERNAL PRESSURE

Substituting $p_i = 0$ into the equations (1), (2), (4), (5), (7), and (8) in Section XXV give:

$$\delta_c' = \delta_c^{\prime M_1} M_1 + \delta_c^{\prime H_1} H_1 - \delta_c^{\prime P_1} p_o \tag{1}$$

$$\theta_c' = \theta_c^{\prime M_1} M_1 + \theta_c^{\prime H_1} H_1 \tag{2}$$

$$\delta_s' = \delta_s^{\prime M_1} M_1 + \delta_s^{\prime H_1} H_1 - \delta_s^{\prime P_1} p_o - \delta_s^{\prime F_1} p_o \tag{3}$$

$$\theta_s' = \theta_s^{\prime M_1} M_1 + \theta_s^{\prime H_1} H_1 - \theta_s^{\prime F_1} p_o \tag{4}$$

$$(\delta_c^{\prime M_1} - \delta_s^{\prime M_1}) M_1 + (\delta_c^{\prime H_1} - \delta_s^{\prime H_1}) H_1 = -(\delta_s^{\prime P_1} - \delta_c^{\prime P_1}) p_o - \delta_s^{\prime F_1} p_o \tag{5}$$

$$(\theta_c^{\prime M_1} - \theta_s^{\prime M_1}) M_1 + (\theta_c^{\prime H_1} - \theta_s^{\prime H_1}) H_1 = -\theta_s^{\prime F_1} p_o \tag{6}$$

XXVIII. THIN-WALLED CYLINDRICAL SHELLS WITH TORISPHERICAL-HEAD CLOSURE UNDER INTERNAL AND EXTERNAL PRESSURE

The analysis of thin-walled cylindrical shell with torispherical head closure (Fig. 1) can be accomplished by knowing the behaviors of three separate parts. Edge deflection and rotation equations can be written in terms of edge influence coefficients and then the condition of continuity may be applied [11,12,13,15,37,42].

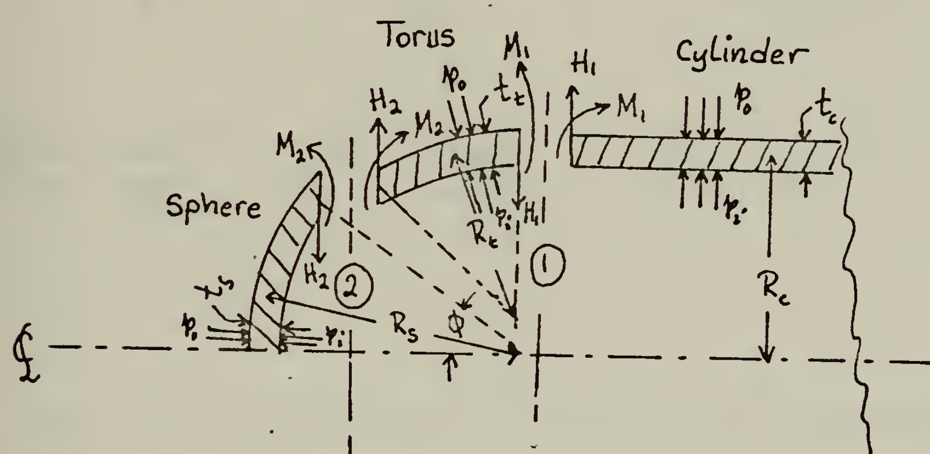


Fig. 1 Semi-infinite cylinder with closure that is partly toroidal and partly spherical (i.e., "torispherical.")

For the cylindrical shell at junction 1 , the deflection and rotation can be written in terms of edge influence coefficients as follows:

$$\delta_c^{\prime} = \delta_c^{\prime M_1} M_1 + \delta_c^{\prime H_1} H_1 + \delta_c^{\prime P_1} (p_i - p_o) \tag{1}$$

$$\theta_c^{\prime} = \theta_c^{\prime M_1} M_1 + \theta_c^{\prime H_1} H_1 \tag{2}$$

The edge influence coefficients may be taken from equation (3) in Section XXV or alternatively can be written as follows:

$$\delta_c^{\prime M_1} = \frac{2(\beta_c)^2}{k_c} \qquad \delta_c^{\prime H_1} = \frac{2\beta_c}{k_c} \qquad \delta_c^{\prime P_1} = \frac{(2-\nu_c)(R_c)^2}{2Et_c}$$

$$\theta_c^{\prime M_1} = \frac{4(\beta_c)^3}{k_c} \qquad \theta_c^{\prime H_1} = \frac{2(\beta_c)^2}{k_c} \qquad \theta_c^{\prime P_1} = 0 \tag{3}$$

For the toroidal shell at junction 1 the deflection and rotation can be written in terms of edge influence coefficients as follows:

$$\delta_t^{\prime} = \delta_t^{\prime M_1} M_1 + \delta_t^{\prime M_2} M_2 + \delta_t^{\prime H_1} H_1 + \delta_t^{\prime H_2} H_2 + \delta_t^{\prime P_1} (p_i - p_o) \tag{4}$$

$$\theta_t^{\prime} = \theta_t^{\prime M_1} M_1 + \theta_t^{\prime M_2} M_2 + \theta_t^{\prime H_1} H_1 + \theta_t^{\prime H_2} H_2 + \theta_t^{\prime P_1} (p_i - p_o) \tag{5}$$

Similarly for junction 2:

$$\delta_t^2 = \delta_t^{2M_1} M_1 + \delta_t^{2M_2} M_2 + \delta_t^{2H_1} H_1 + \delta_t^{2H_2} H_2 + \delta_t^{2P_2} (p_i - p_o) \tag{6}$$

$$\theta_t^2 = \theta_t^{2M_1} M_1 + \theta_t^{2M_2} M_2 + \theta_t^{2H_1} H_1 + \theta_t^{2H_2} H_2 + \theta_t^{2P_2} (p_i - p_o) \tag{7}$$

In the toroidal shell it is not permissible to assume that its two edges are so remote from one another that their mutual interaction may be neglected; accordingly the formulas for the edge influence coefficients are rather complicated and will not be reported here. Necessary information may be found in [11,12]. However, as before, the cylinder may be assumed to be of semi-infinite length and the spherical portion is assumed to subtend a substantial angle, (i.e., $\phi^\circ > \frac{162}{\lambda_s}$).

For the spherical shell at junction 2, the deflection and the rotation can be written in terms of the edge influence coefficients as follows:

$$\delta_s^2 = \delta_s^{2M_2} M_2 + \delta_s^{2H_2} H_2 + \delta_s^{2P_2} (p_i - p_o) \tag{8}$$

$$\theta_s^2 = \theta_s^{2M_2} M_2 + \theta_s^{2H_2} H_2 \tag{9}$$

The edge influence coefficients can be written as follows:

$$\delta_s^{2M_2} = \frac{2(\lambda_s)^2 \sin\phi}{E t_s \chi_1} \qquad \delta_s^{2H_2} = -\frac{\lambda_s R_s (\sin\phi)^2}{E t_s}\left(\chi_2 + \frac{1}{\chi_1}\right)$$

$$\delta_s^{2P_2} = \frac{(1-\nu_s)(R_s)^2 \sin\phi}{2 E t_s} \tag{10}$$

$$\theta_s^{2M_2} = -\frac{4(\lambda_s)^2}{E R_s t_s \chi_1} \qquad \theta_s^{2H_2} = \frac{2(\lambda_s)^2 \sin\phi}{E t_s \chi_1} \qquad \theta_s^{2P_2} = 0$$

where

$$\chi_1 = 1 - \frac{1-2\nu_s}{2\lambda_s}\cot\phi$$

$$\chi_2 = 1 - \frac{1+2\nu_s}{2\lambda_s}\cot\phi$$

Conditions of continuity give four simultaneous equations to solve, as follows:

$$\delta_c^{1M_1} M_1 + \delta_c^{1H_1} H_1 + \delta_c^{1P_1}(p_i - p_o) = \delta_t^{1M_1} M_1 + \delta_t^{1M_2} M_2 + \delta_t^{1H_1} H_1 + \delta_t^{1H_2} H_2 + \delta_t^{1P_1}(p_i - p_o) \tag{11}$$

$$\delta_s^{2M_2} M_2 + \delta_s^{2H_2} H_2 + \delta_s^{2P_2} (p_i - p_o) = \delta_t^{2M_1} M_1 + \delta_t^{2M_2} M_2 + \delta_t^{2H_1} H_1 + \delta_t^{2H_2} H_2 + \delta_t^{2P_2} (p_i - p_o) \quad (12)$$

$$\theta_c^{1M_1} M_1 + \theta_c^{1H_1} H_1 = \theta_t^{1M_1} M_1 + \theta_t^{1M_2} M_2 + \theta_t^{1H_1} H_1 + \theta_t^{1H_2} H_2 + \theta_t^{1P_1} (p_{i} - p_o) \quad (13)$$

$$\theta_s^{2M_2} M_2 + \theta_s^{2H_2} H_2 = \theta_t^{2M_1} M_1 + \theta_t^{2M_2} M_2 + \theta_t^{2H_1} H_1 + \theta_t^{2H_2} H_2 + \theta_t^{2P_2} (p_{i} - p_o) \quad (14)$$

Simultaneous solution of equations (11), (12), (13) and (14) gives the edge loads.

XXIX. THIN-WALLED CYLINDRICAL SHELLS WITH TORISPHERICAL-HEAD CLOSURE UNDER INTERNAL PRESSURE

Substituting $p_o = 0$ into the appropriate equations in Section XXVIII gives

$$\delta_c' = \delta_c^{1M_1} M_1 + \delta_c^{1H_1} H_1 + \delta_c^{1P_1} p_{i}, \quad (1)$$

$$\theta_c' = \theta_c^{1M_1} M_1 + \theta_c^{1H_1} H_1 \quad (2)$$

$$\delta_t^1 = \delta_t^{1M_1} M_1 + \delta_t^{1M_2} M_2 + \delta_t^{1H_1} H_1 + \delta_t^{1H_2} H_2 + \delta_t^{1P_1} P_2. \tag{3}$$

$$\theta_t^1 = \theta_t^{1M_1} M_1 + \theta_t^{1M_2} M_2 + \theta_t^{1H_1} H_1 + \theta_t^{1H_2} H_2 + \theta_t^{1P_1} P_2. \tag{4}$$

$$\delta_t^2 = \delta_t^{2M_1} M_1 + \delta_t^{2M_2} M_2 + \delta_t^{2H_1} H_1 + \delta_t^{2H_2} H_2 + \delta_t^{2P_2} P_2. \tag{5}$$

$$\theta_t^2 = \theta_t^{2M_1} M_1 + \theta_t^{2M_2} M_2 + \theta_t^{2H_1} H_1 + \theta_t^{2H_2} H_2 + \theta_t^{2P_2} P_2. \tag{6}$$

$$\delta_s^2 = \delta_s^{2M_2} M_2 + \delta_s^{2H_2} H_2 + \delta_s^{2P_2} P_2. \tag{7}$$

$$\theta_s^2 = \theta_s^{2M_2} M_2 + \theta_s^{2H_2} H_2 \tag{8}$$

$$\delta_c^{1M_1} M_1 + \delta_c^{1H_1} H_1 + \delta_c^{1P_1} P_2 = \delta_t^{1M_1} M_1 + \delta_t^{1M_2} M_2 + \delta_t^{1H_1} H_1 + \delta_t^{1H_2} H_2 + \delta_t^{1P_1} P_2. \tag{9}$$

$$\delta_s^{2M_2} M_2 + \delta_s^{2H_2} H_2 + \delta_s^{2P_2} P_2 = \delta_t^{2M_1} M_1 + \delta_t^{2M_2} M_2 + \delta_t^{2H_1} H_1 + \delta_t^{2H_2} H_2 + \delta_t^{2P_2} P_2. \tag{10}$$

$$\theta_c^{1M_1} M_1 + \theta_c^{1H_1} = \theta_t^{1M_1} M_1 + \theta_t^{1M_2} M_2 + \theta_t^{1H_1} H_1 + \theta_t^{1H_2} H_2 + \theta_t^{1P_1} p_i. \tag{11}$$

$$\theta_s^{2M_2} M_2 + \theta_s^{2H_2} H_2 = \theta_t^{2M_1} M_1 + \theta_t^{2M_2} M_2 + \theta_t^{2H_1} H_1 + \theta_t^{2H_2} H_2 + \theta_t^{2P_2} p_i. \tag{12}$$

XXX. THIN-WALLED CYLINDRICAL SHELLS WITH TORISPHERICAL HEAD CLOSURE UNDER EXTERNAL PRESSURE

Substituting $p_i = 0$ into the appropriate equation in Section XXVIII gives

$$\delta_c^1 = \delta_c^{1M_1} M_1 + \delta_c^{1H_1} H_1 - \delta_c^{1P_1} p_o \tag{1}$$

$$\theta_c^1 = \theta_c^{1M_1} M_1 + \theta_c^{1H_1} H_1 \tag{2}$$

$$\delta_t^1 = \delta_t^{1M_1} M_1 + \delta_t^{1M_2} M_2 + \delta_t^{1H_1} H_1 + \delta_t^{1H_2} H_2 - \delta_t^{1P_1} p_o \tag{3}$$

$$\theta_t^1 = \theta_t^{1M_1} M_1 + \theta_t^{1M_2} M_2 + \theta_t^{1H_1} H_1 + \theta_t^{1H_2} H_2 - \theta_t^{1P_1} p_o \tag{4}$$

$$\delta_t^2 = \delta_t^{2M_1} M_1 + \delta_t^{2M_2} M_2 + \delta_t^{2H_1} H_1 + \delta_t^{2H_2} H_2 - \delta_t^{2P_2} p_0 \tag{5}$$

$$\theta_t^2 = \theta_t^{2M_1} M_1 + \theta_t^{2M_2} M_2 + \theta_t^{2H_1} H_1 + \theta_t^{2H_2} H_2 - \theta_t^{2P_2} p_0 \tag{6}$$

$$\delta_s^2 = \delta_s^{2M_2} M_2 + \delta_s^{2H_2} H_2 - \delta_s^{2P_2} p_0 \tag{7}$$

$$\theta_s^2 = \theta_s^{2M_2} M_2 + \theta_s^{2H_2} H_2 \tag{8}$$

$$\delta_c^{1M_1} M_1 + \delta_c^{1H_1} H_1 - \delta_c^{1P_1} p_0 = \delta_t^{1M_1} M_1 + \delta_t^{1M_2} M_2 + \delta_t^{1H_1} H_1 + \delta_t^{1H_2} H_2 - \delta_t^{1P_1} p_0 \tag{9}$$

$$\delta_s^{2M_2} M_2 + \delta_s^{2H_2} H_2 - \delta_s^{2P_2} p_0 = \delta_t^{2M_1} M_1 + \delta_t^{2M_2} M_2 + \delta_t^{2H_1} H_1 + \delta_t^{2H_2} H_2 - \delta_t^{2P_2} p_0 \tag{10}$$

$$\theta_c^{1M_1} M_1 + \theta_c^{1H_1} H_1 = \theta_t^{1M_1} M_1 + \theta_t^{1M_2} M_2 + \theta_t^{1H_1} H_1 + \theta_t^{1H_2} H_2 - \theta_t^{1P_1} p_0 \tag{11}$$

$$\theta_s^{2M_2} M_2 + \theta_s^{2H_2} H_2 = \theta_t^{2M_1} M_1 + \theta_t^{2M_2} M_2 + \theta_t^{2H_1} H_1 + \theta_t^{2H_2} H_2 - \theta_t^{2P_2} p_0 \tag{12}$$

XXXI. THIN-WALLED CYLINDRICAL SHELLS WITH ELLIPSOID-TORUS TRANSITION UNDER INTERNAL PRESSURE

Analysis can be performed by writing the deflection and rotation equations for each junction in terms of the edge influence coefficients. Then the condition of continuity can be applied to obtain a set of simultaneous equations [13]. Two cylindrical shells connected by an ellipsoidal-torus transition have been taken as a typical example (Fig. 1).

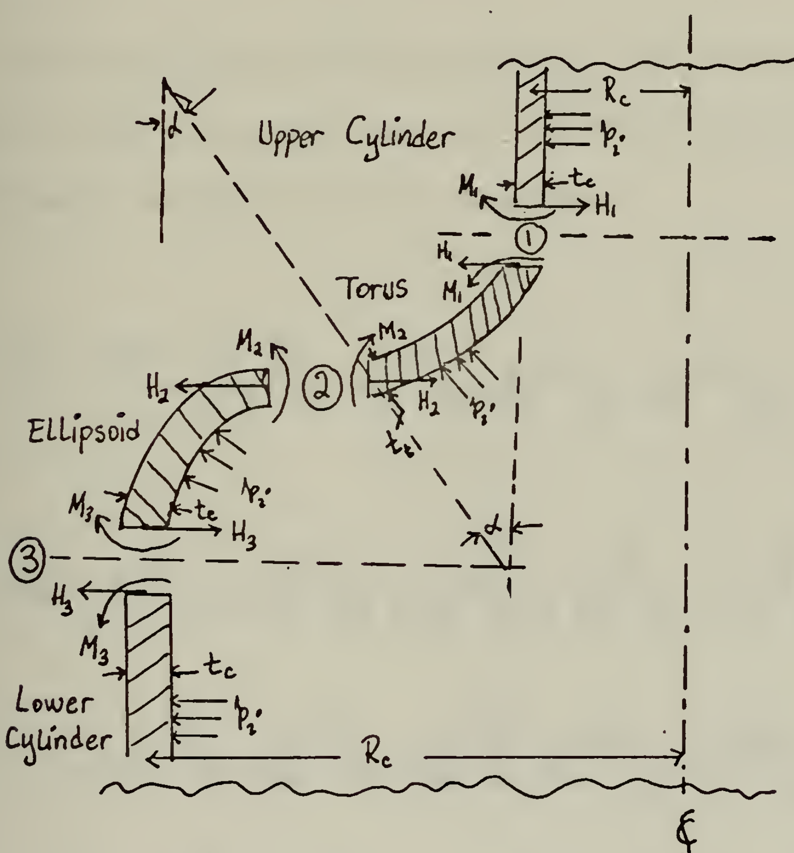


Fig. 1 Ellipsoid-torus transition between coaxial cylinders

For upper cylindrical shell the deflection and rotation equations at junction (1), can be written in terms of edge influence coefficients as follows:

$$\delta_c^1 = \delta_c^{1M_1} M_1 + \delta_c^{1H_1} H_1 + \delta_c^{1P_1} p_2 . \qquad (1)$$

$$\theta_c^1 = \theta_c^{1M_1} M_1 + \theta_c^{1H_1} H_1 \qquad (2)$$

For the torus the deflections and rotations at junctions (1) and (2) respectively, can be written in terms of edge influence coefficients as follows:

$$\delta_t^1 = \delta_t^{1M_1} M_1 + \delta_t^{1M_2} M_2 + \delta_t^{1H_1} H_1 + \delta_t^{1H_2} H_2 + \delta_t^{1P_1} p_2 . \qquad (3)$$

$$\theta_t^1 = \theta_t^{1M_1} M_1 + \theta_t^{1M_2} M_2 + \theta_t^{1H_1} H_1 + \theta_t^{1H_2} H_2 + \theta_t^{1P_1} p_2 . \qquad (4)$$

$$\delta_t^2 = \delta_t^{2M_1} M_1 + \delta_t^{2M_2} M_2 + \delta_t^{2H_1} H_1 + \delta_t^{2H_2} H_2 + \delta_t^{2P_2} p_2 . \qquad (5)$$

$$\theta_t^2 = \theta_t^{2M_1} M_1 + \theta_t^{2M_2} M_2 + \theta_t^{2H_1} H_1 + \theta_t^{2H_2} H_2 + \theta_t^{2P_2} p_2 . \qquad (6)$$

For ellipsoidal shell the deflection and rotation at junction (2) and (3) respectively, can be written in terms of edge influence coefficients as follows:

$$\delta_e^2 = \delta_e^{2M_2} M_2 + \delta_e^{2M_3} M_3 + \delta_e^{2H_2} H_2 + \delta_e^{2H_3} H_3 + \delta_e^{2P_2} p_2. \tag{7}$$

$$\theta_e^2 = \theta_e^{2M_2} M_2 + \theta_e^{2M_3} M_3 + \theta_e^{2H_2} H_2 + \theta_e^{2H_3} H_3 + \theta_e^{2P_2} p_2. \tag{8}$$

$$\delta_e^3 = \delta_e^{3M_2} M_2 + \delta_e^{3M_3} M_3 + \delta_e^{3H_2} H_2 + \delta_e^{3H_3} H_3 + \delta_e^{3P_3} p_2. \tag{9}$$

$$\theta_e^3 = \theta_e^{3M_2} M_2 + \theta_e^{3M_3} M_3 + \theta_e^{3H_2} H_2 + \theta_e^{3H_3} H_3 + \theta_e^{3P_3} p_2. \tag{10}$$

For the lower cylindrical shell the deflection and rotation at junction (3) can be written in terms of the edge influence coefficients as follows:

$$\delta_c^3 = \delta_c^{3M_3} M_3 + \delta_c^{3H_3} H_3 + \delta_c^{3P_3} p_2. \tag{11}$$

$$\theta_c^3 = \theta_c^{3M_3} M_3 + \theta_c^{3H_3} H_3 \tag{12}$$

Conditions of continuity give the following simultaneous equations:

$$(\delta_c^{1M_1} - \delta_t^{1M_1})M_1 - \delta_t^{1M_2}M_2 + (\delta_c^{1H_1} - \delta_t^{1H_1})H_1 - \delta_t^{1H_2}H_2 + (\delta_c^{1P_1} - \delta_t^{1P_1})\,p_2^{\circ} = 0 \qquad (13)$$

$$\delta_t^{2M_1}M_1 + (\delta_t^{2M_2} - \delta_e^{2M_2})M_2 - \delta_e^{2M_3}M_3 + \delta_t^{2H_1}H_1 + (\delta_t^{2H_2} - \delta_e^{2H_2})H_2 - \delta_e^{2H_3}H_3 + (\delta_t^{2P_2} - \delta_e^{2P_2})\,p_2^{\circ} = 0 \qquad (14)$$

$$\delta_e^{3M_2}M_2 + (\delta_e^{3M_3} - \delta_c^{3M_3})M_3 + \delta_e^{3H_2}H_2 + (\delta_e^{3H_3} - \delta_c^{3H_3})H_3 + (\delta_e^{3P_3} - \delta_c^{3P_3})\,p_2^{\circ} = 0 \qquad (15)$$

$$(\theta_c^{1M_1} - \theta_t^{1M_1})M_1 - \theta_t^{1M_2}M_2 + (\theta_c^{1H_1} - \theta_t^{1H_1})H_1 - \theta_t^{1H_2}H_2 - \theta_t^{1P_1}\,p_2^{\circ} = 0 \qquad (16)$$

$$\theta_t^{2M_1}M_1 + (\theta_t^{2M_2} - \theta_e^{2M_2})M_2 - \theta_e^{2M_3} + M_3 + \theta_t^{2H_1}H_1 + (\theta_t^{2H_2} - \theta_e^{2H_2})H_2 - \theta_e^{2H_3}H_3 + (\theta_t^{2P_2} - \theta_e^{2P_2})\,p_2^{\circ} = 0 \qquad (17)$$

$$\theta_e^{3M_2}M_2 + (\theta_e^{3M_3} - \theta_c^{3M_3})M_3 + \theta_e^{3H_2}H_2 + (\theta_e^{3H_3} - \theta_c^{3H_3})H_3 + \theta_e^{3P_3}\,p_2^{\circ} = 0 \qquad (18)$$

Tabulated numerical values of these influence coefficients may be found in [13] and in references cited in [13]. Close attention should be given to the sign conventions employed in all these references, including [12].

Simultaneous solution of equations (13), (14), (15), (16), (17) and (18) gives the edge loads.

If the toroidal and ellipsoidal shell segments are "long", that is if their meridional arc lengths are several times the local value of an attenuation length, then the "cross-influence" coefficients (reflecting the effect of conditions at one edge upon those at the other) are quite small compared to the "self-influence" coefficients and may be neglected. This affords a considerable simplification, as follows:

$$\delta_t^1 = \delta_t^{1M_1} M_1 + \delta_t^{1H_1} H_1 + \delta_t^{1P_1} p_2. \tag{19}$$

$$\theta_t^1 = \theta_t^{1M_1} M_1 + \theta_t^{1H_1} H_1 + \theta_t^{1P_1} p_2, \tag{20}$$

$$\delta_t^2 = \delta_t^{2M_2} M_2 + \delta_t^{2H_2} H_2 + \delta_t^{2P_2} p_2. \tag{21}$$

$$\theta_t^2 = \theta_t^{2M_2} M_2 + \theta_t^{2H_2} H_2 + \theta_t^{2P_2} p_2, \tag{22}$$

$$\delta_e^2 = \delta_e^{2M_2} M_2 + \delta_e^{2H_2} H_2 + \delta_e^{2P_2} p_2. \tag{23}$$

$$\theta_e^2 = \theta_e^{2M_2} M_2 + \theta_e^{2H_2} H_2 + \theta_e^{2P_2} P_{2^\circ} \tag{24}$$

$$\delta_e^3 = \delta_e^{3M_3} M_3 + \delta_e^{3H_3} H_3 + \delta_e^{3P_3} P_{2^\circ} \tag{25}$$

$$\theta_e^3 = \theta_e^{3M_3} M_3 + \theta_e^{3H_3} H_3 + \theta_e^{3P_3} P_{2^\circ} \tag{26}$$

Then the simultaneous equations (13), (14), (15), (16), (17), and (18) become:

$$(\delta_c^{1M_1} - \delta_t^{1M_1}) M_1 + (\delta_c^{1H_1} - \delta_t^{1H_1}) H_1 + (\delta_c^{1P_1} - \delta_t^{1P_1}) P_{2^\circ} = 0 \tag{27}$$

$$(\delta_t^{2M_2} - \delta_e^{2M_2}) M_2 + (\delta_t^{2H_2} - \delta_e^{2H_2}) H_2 + (\delta_t^{2P_2} - \delta_e^{2P_2}) P_{2^\circ} = 0 \tag{28}$$

$$(\delta_e^{3M_3} - \delta_c^{3M_3}) M_3 + (\delta_e^{3H_3} - \delta_c^{3H_3}) H_3 + (\delta_e^{3P_3} - \delta_c^{3P_3}) P_{2^\circ} = 0 \tag{29}$$

$$(\theta_c^{1M_1} - \theta_t^{1M_1}) M_1 + (\theta_c^{1H_1} - \theta_t^{1H_1}) H_1 - \theta_t^{1P_1} P_{2^\circ} = 0 \tag{30}$$

$$(\theta_t^{2M_2} - \theta_e^{2M_2})M_2 + (\theta_t^{2H_2} - \theta_e^{2H_2})H_2 + (\theta_t^{2P_2} - \theta_e^{2P_2})\,p_{2^\circ} = 0 \tag{31}$$

$$(\theta_e^{3M_3} - \theta_c^{3M_3})M_3 + (\theta_e^{3H_3} - \theta_c^{3H_3})H_3 + \theta_e^{3P_3}\,p_{2^\circ} \tag{32}$$

These equations are not only simplified in appearance but are now uncoupled pairwise so that (27) and (30) yield M_1 and H_1, (28) and (31) yield M_2 and H_2, and (29) and (32) yield M_3 and H_3.

The reader may observe that in this section an ellipsoidal segment is involved although we have not elsewhere given a treatment for a full ellipsoidal closure. Data for influence coefficients of so-called "open crown" ellipsoidal shells is more readily available than for complete semi-ellipsoidal closures. See Galletly [13] and references cited therein. Kraus et al. [22] have used a digital computer to analyze thin-walled cylindrical vessels with full semi-elliptical heads but the results are not in the form of influence coefficients. Instead, coefficients of (stress ÷ pressure) are tabulated for useful ranges of several geometric parameters describing the configuration. It is not in the spirit of this thesis to reproduce this tabular data and the reader is referred to [22].

XXXII. AXIAL STRESS IN THIN-WALLED CYLINDRICAL SHELLS UNDER INTERNAL AND EXTERNAL PRESSURE TOGETHER WITH AXIALLY APPLIED FORCE

Uniform distribution of the axial stress on the cylinder cross section has been assumed. The axial stress is then a function of the geometry of the cylinder [6,32,33,34].

Axial stress can be written as follows:

$$\sigma_x = \frac{R_c}{2t_c}(p_{i^\circ} - p_o) + \frac{q}{t_c} \quad (1)$$

(Note that $F = 2\pi R_c q$ is externally applied axial tensile force)

If only internal pressure is applied equation (1) becomes

$$\sigma_x = \frac{R_c}{4t_c} p_{i^\circ} + \frac{q}{t_c} \quad (2)$$

If only external pressure is applied equation (1) becomes

$$\sigma_x = -\frac{R_c}{4t_c} p_o + \frac{q}{t_c} \quad (3)$$

XXXIII. AXIAL STRESS IN CLOSED THICK-WALLED CYLINDRICAL PRESSURE VESSELS UNDER INTERNAL AND EXTERNAL PRESSURE

Axial stress is uniformaly distributed on the cylinder cross section [6,32,33,34] except near the junction of cylinder and closure. If there is no axial loading other than that due to pressure the axial stress can be written as follows:

$$\sigma_x = \frac{(R_{ci})^2 p_i - (R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} \qquad (1)$$

If only internal pressure is applied equation (1) becomes

$$\sigma_x = \frac{(R_{ci})^2 p_i}{(R_{co})^2 - (R_{ci})^2} \qquad (2)$$

If only external pressure is applied equation (1) becomes

$$\sigma_x = - \frac{(R_{co})^2 p_o}{(R_{co})^2 - (R_{ci})^2} \qquad (3)$$

XXXIV. AXIAL STRESS IN THICK-WALLED CYLINDRICAL PRESSURE VESSELS UNDER INTERNAL AND EXTERNAL PRESSURE TOGETHER WITH AXIALLY APPLIED FORCE

In this case the axial stress equations can be written as follows:

$$\sigma_x = \frac{(R_{ci})^2 p_i - (R_{co})^2 p_o + \frac{F}{\pi}}{(R_{co})^2 - (R_{ci})^2} \tag{1}$$

where F is the externally applied axial tensile force.

If only internal pressure is applied equation (1) becomes

$$\sigma_x = \frac{(R_{ci})^2 p_i + \frac{F}{\pi}}{(R_{co})^2 - (R_{ci})^2} \tag{2}$$

If only external pressure is applied equation (1) becomes

$$\sigma_x = - \frac{(R_{co})^2 p_o - \frac{F}{\pi}}{(R_{co})^2 - (R_{ci})^2} \tag{3}$$

XXXV. BUCKLING OF THIN-WALLED CYLINDRICAL SHELLS WITH OPEN ENDS UNDER EXTERNAL PRESSURE

The analysis is discussed in [4,8,15,33,35,37,42].

Only the results will be given here. It is assumed that there is no restraint against axial deformation and that the only restraint at the ends is agains radial deformation.

The general solution can be written as follows

$$p_{cr} = \frac{1}{3}\left[(n)^2 - 1 + \frac{2(n)^2 - 1 - \nu_c}{(n)^2\left(\dfrac{2\ell}{d_{co}\pi}\right)^2 - 1}\right] \times \frac{2E}{(1-\nu_c^2)} \times \frac{(t_c)^3}{(d_{co})^3} + \frac{2E \times \dfrac{t_c}{d_{co}}}{\left[(n)^2 - 1\right]\left[(n)^2\left(\dfrac{2\ell}{d_{co}\pi}\right)^2 + 1\right]^2} \tag{1}$$

where n is an integer which must be selected (by trial or otherwise) so as to minimize the value of this expression. The value of n, however, must be not less than 2.

If the ratio d_{c_o}/ℓ is very small, the expression simplifies considerably and the value n = 2 is then seen to be appropriate so that,

$$p_{cr} = \frac{E(t_c)^3}{4(1-\nu_c^2)(R_c)^3} \tag{2}$$

The preceding equation presumes elastic action. The following formula is for elastic action with no strain hardening [33].

$$p_{cr} = \frac{t_c}{R_c} \times \frac{\sigma_{y.p.}}{1 + 4 \times \dfrac{(R_c)^2}{E(t_c)^2} \times \sigma_{y.p.}} \tag{3}$$

XXXVI. BUCKLING OF THIN-WALLED CYLINDRICAL SHELLS WITH CLOSED ENDS UNDER EXTERNAL PRESSURE

General buckling pressure can be written as follows [4, 33]:

$$p_{cr} = \left\{ \frac{1}{3} \left[(n)^2 + \left(\frac{d_{co}\pi}{2\ell} \right)^2 \right]^2 \times \frac{2E}{1-\nu_c^2} \times \frac{(t_c)^3}{(d_{co})^3} + \frac{2E \times \frac{t_c}{d_{co}}}{\left[(n)^2 \left(\frac{2\ell}{d_{co}\pi} \right)^2 + 1 \right]^2} \right\} \times \frac{1}{(n)^2 + \frac{1}{2}\left(\frac{d_{co}\pi}{2\ell} \right)^2} \tag{1}$$

n is an integer which must be selected (by trial or otherwise) so as to minimize the value of this expression.

The end constraints are as in Section XXXV except that an axially compressive force $P = p(d_{c_o})^2/4$ is also present. Formula (1) is cumbersome to use, so that the following approximate formula in which n, the number of lobes of the buckled shape does not appear, has also been employed in this case

$$p_{cr} = \frac{2.42\,E}{(1-\nu_c^2)^{\frac{3}{4}}} \times \frac{\left(\frac{t_c}{d_{co}} \right)^{\frac{5}{2}}}{\frac{\ell}{d_{co}} - 0.45 \left(\frac{t_c}{d_{co}} \right)^{\frac{1}{2}}} \tag{2}$$

XXXVII. SYMMETRICAL BUCKLING OF THIN-WALLED CYLINDRICAL SHELLS UNDER ACTION OF UNIFORM AXIAL COMPRESSION

The configuration of the system is shown in Fig. 1.

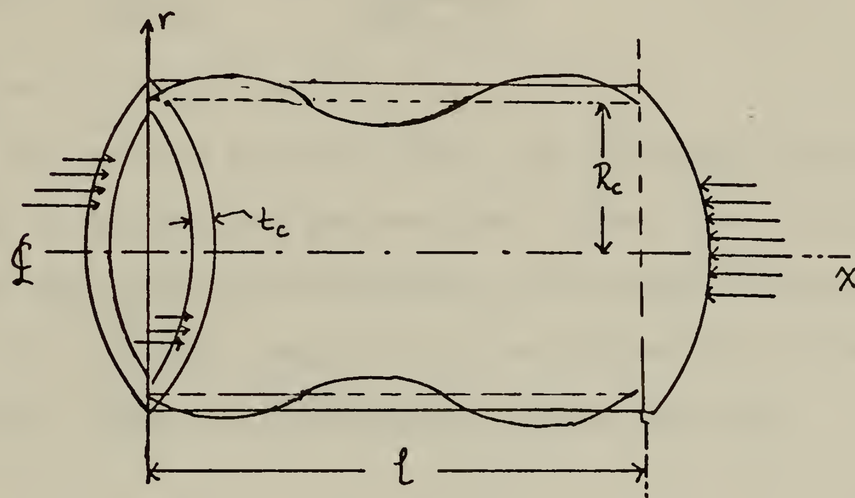


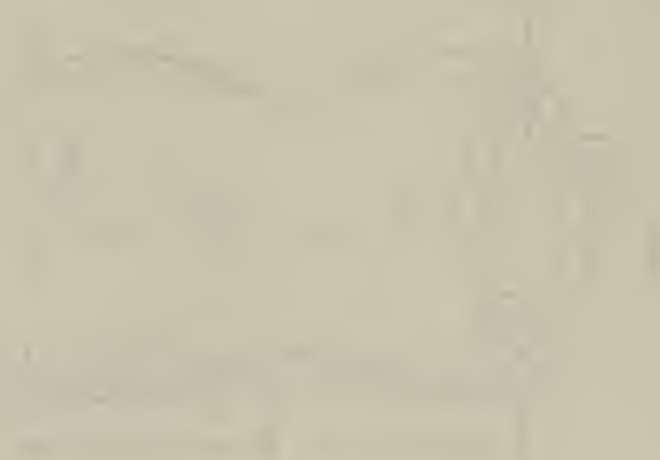
Fig. 1 Buckling of thin-walled cylinder due to due to uniform axial compression

The critical axial compressive stress is

$$\sigma_{cr} = \frac{E t_c}{R_c}\left[\frac{(\ell)^2}{(n\pi)^2 R_c t_c} + \frac{(n\pi)^2 R_c t_c}{12(1-\nu_c^2)(\ell)^2}\right] \tag{1}$$

where n is an integer ($n \geq 1$) which must be chosen so as to minimize this expression; n represents the number of half sine waves into which the cylinder buckles. The half wave-length is ℓ/n. The value of n is given approximately by

$$n = \frac{\ell \beta_c}{\pi} \tag{2}$$



and if this value is substituted in Equation (1) the following approximation is obtained

$$\sigma_{cr} = \frac{2E}{(\beta_c R_c)^2} = \frac{\frac{Et_c}{R_c}}{\sqrt{3(1-\nu^2)}} \tag{3}$$

The case n = 3 is shown in Fig. 1.

When the shell is not very thin and buckling occurs at a stress that is beyond the proportional limit the critical load can be obtained by introducing the tangent modulus, E_t, instead of the elastic modulus E at an appropriate place in the analysis. Then the approximate result becomes

$$\sigma_{cr} = \frac{t_c \sqrt{E E_t}}{R_c \sqrt{3(1-\nu_c^2)}} \tag{4}$$

If the thin-walled cylindrical shell is short, i.e., if

$$\ell < \frac{\pi}{\sqrt{2}\,\beta_c}$$

the critical buckling stress can be written as follows:

$$\sigma_{cr} = \frac{(\pi)^2 E (t_c)^2}{12(1-\nu_c^2)(\ell)^2} \tag{5}$$

XXXVIII. BUCKLING OF LONG THIN-WALLED CYLINDRICAL SHELLS SUBJECTED TO TORSION

Thin cylindrical shells may buckle as a whole without flattening of the circular cross section, or may buckle locally with modification of cross section. In the latter case if the shell is long enough, the critical moment does not depend essentially upon the conditions of end restraint, and is given by [35]

$$M_{cr} = \frac{\pi \sqrt{2} \cdot E \cdot \sqrt{R_c (t_c)^5}}{3 \cdot \sqrt[4]{(1-\nu_c^2)^3}} \tag{1}$$

From the simple relation

$$M_{cr} = 2\pi (R_c)^2 t_c \tau_{cr} \tag{2}$$

we obtain

$$\tau_{cr} = \frac{E \cdot \sqrt{\left(\frac{t_c}{R_c}\right)^3}}{3\sqrt{2} \cdot \sqrt[4]{(1-\nu_c^2)^3}} \tag{3}$$

For shorter cylinders, the modification of cross section involves several (i.e., more than two) lobes and the formulas in Section XXXIX and XL should be used.

However, the cylinder may also buckle as a whole. If no lateral restraining moments are applied at the ends, the corresponding critical moment is given by

$$M_{cr} = \frac{2(\pi)^2 (R_c)^3 t_c E}{\ell} \tag{4}$$

which may be smaller than the result given by Eq. (1) of the present section or the analysis of Sections XXXIX or XL, whichever may otherwise be applicable. Similar results are not available for other than laterally momentless end conditions.

XXXIX. BUCKLING OF THIN-WALLED SHORT CYLINDRICAL SHELLS WITH CLAMPED EDGES SUBJECTED TO TORSION

If [35],

$$\ell < 11.2 \times \sqrt[4]{(1-\nu_c^2)} \times R_c \times \sqrt{\frac{R_c}{2t_c}} \tag{1}$$

then the critical shear stress can be obtained from

$$\frac{(1-\nu_c^2)}{E}\left(\frac{\ell}{t_c}\right)^2 \tau_{cr} = 4.6 + \sqrt{7.8 + 1.67\left[\sqrt{1-\nu_c^2} \times \frac{(\ell)^2}{2R_c t_c}\right]^{3/2}} \tag{2}$$

and the corresponding value of M_{cr} may be obtained from Equation 2 of Section XXXVII. Cf. also Equation 4 of Section XXXVIII.

XL. BUCKLING OF THIN-WALLED SHORT CYLINDRICAL SHELLS WITH SIMPLY SUPPORTED EDGES SUBJECTED TO TORSION

If [35],

$$\ell < 9.4 \times \sqrt[4]{1-\nu^2} \times R_c \times \sqrt{\frac{R_c}{2t_c}} \tag{1}$$

then the critical shear stress can be obtained from

$$\frac{(1-\nu^2)}{E}\left(\frac{\ell}{t_c}\right)^2 \tau_{cr} = 4.39 \times \sqrt{1+0.0257 \times \sqrt[4]{(1-\nu^2)^3} \times \left[\frac{\ell}{\sqrt{R_c t_c}}\right]^3} \tag{2}$$

and the corresponding value of M_{cr} may be obtained from Equation 2 of Section XXXVIII. Cf. also Equation 4 of Section XXXVIII.

XLI. BUCKLING OF UNIFORMLY COMPRESSED THIN-WALLED SPHERICAL SHELLS UNDER EXTERNAL PRESSURE

The theoretical critical buckling pressure and stress are [35]

$$p_{cr} = \frac{2Et_s}{(1-\nu_s^2)R_s}\left[\sqrt{\frac{1-\nu_s^2}{3}} \times \frac{t_s}{R_s} - \frac{\nu_s (t_s)^2}{2(R_s)^2}\right] \cong \frac{2E(t_s)^2}{\sqrt{3(1-\nu_s^2)}\,(R_s)^2} \tag{1}$$

$$\sigma_{cr} = \frac{E t_s}{\sqrt{3(1-\nu_s^2)} \times R_s} \tag{2}$$

However, externally pressurized thin spherical shells seem to be quite sensitive to the presence of slight imperfections and to small disturbances during loading, and observed values of critical pressure are usually much smaller than given by Eq. (1). Accordingly, the following empirical formula is offered [35]

$$p_{cr} = 0.247\left(1 - \frac{\Phi^\circ}{100}\right)\left(1 - 0.000175\,\frac{R_s}{t_s}\right)E_x\left(\frac{t_s}{R_s}\right)^2 \tag{3}$$

which agrees with experiments for

$$400 \leq \frac{R_c}{t_c} \leq 2000 \qquad \text{and} \qquad 20^\circ \leq \phi^\circ \leq 60^\circ$$

See Fig. 1.

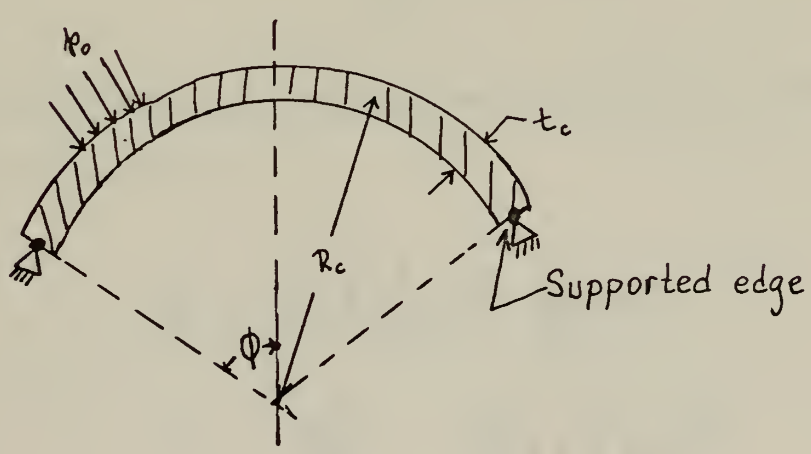


Fig. 1 Spherical dome under external pressure

XLII. THIN-WALLED CYLINDRICAL SHELLS UNDER CENTRIFUGAL FORCE FIELD

When the cylindrical shell is rotated about its axis with constant angular velocity ω (radians per second) uniform radial loading is produced [33,34]. By considering radial equilibrium of an element cut from a cylinder and assuming that there is no axial load or stress, one obtains

$$\sigma_t = \rho (R_c)^2 (\omega)^2 \tag{1}$$

The radial deflection is

$$\delta_c = \frac{\rho}{E} (R_c)^3 (\omega)^2 \tag{2}$$

and the axial unit strain is

$$\varepsilon_x = - \frac{\nu_c \rho}{E} (R_c)^2 (\omega)^2 \tag{3}$$

If $\sigma_x \neq 0$, the following formulas should be used in place of (2) and (3)

$$\delta_c = \frac{\rho}{E} (R_c)^3 (\omega)^2 - \frac{\nu_c}{E} R_c \sigma_x \tag{4}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu \rho}{E} (R_c)^2 (\omega)^2 \tag{5}$$

XLIII. THICK-WALLED CYLINDRICAL PRESSURE VESSELS UNDER CENTRIFUGAL FORCE FIELD

This case is similar to but not the same as the well known case of a rotating thin disk. The latter is a case of plane stress whereas we are here concerned with a case of plane strain.

Specializing first to the case $\varepsilon_x = 0$, it may be shown that the formulas for σ_r and σ_t in the case of a thin disk may be used if the angular velocity squared, ω^2, appearing in the formulas, is replaced by a modified value, namely

$(\bar{\omega})^2 = (\omega)^2 \dfrac{(1 - 2\nu)}{(1 - \nu)^2}$. The resulting formulas are

$$\sigma_r = B\left[(R_{co})^2 + (R_{ci})^2 - \frac{(R_{co})^2 (R_{ci})^2}{(r)^2} - (r)^2\right] \tag{1}$$

$$\sigma_t = B\left[(R_{co})^2 + (R_{ci})^2 + \frac{(R_{co})^2 (R_{ci})^2}{(r)^2} - \frac{1+3\nu}{3+\nu}\cdot(r)^2\right] \tag{2}$$

$$\sigma_x = 2\nu B\left[(R_{co})^2 + (R_{ci})^2 - 2\cdot\frac{1+\nu}{3+\nu}\cdot(r)^2\right] \tag{3}$$

where

$$B = \frac{(3+\nu)(1+\nu)(1-2\nu)\,\rho\,(\omega)^2}{8(1-\nu)} \tag{4}$$

The formula for σ_x comes from the condition $\varepsilon_x = 0$; namely, $\sigma_x = \nu(\sigma_r + \sigma_t)$.

$$(\sigma_r)_{max} = B(R_{co} - R_{ci})^2 \quad \text{at} \quad r = \sqrt{R_{co} R_{ci}} \tag{5}$$

$$(\sigma_t)_{max} = B\left[2 \times \frac{(1-\nu)}{3+\nu} \cdot (R_{ci})^2 + (R_{co})^2\right] \quad \text{at} \quad r = R_{ci} \tag{6}$$

The radial deformation is

$$u = (1-\nu^2)\left(\frac{B}{E}\right)\left\{\left[\frac{(R_{ci})^2 + (R_{co})^2}{1+\nu}\right] r + \frac{(R_{ci})^2 (R_{co})^2}{1-\nu} \times \frac{1}{r} - \frac{1}{3+\nu} \times (r)^3\right\} \tag{7}$$

Thus, also

$$\delta_{ci} = \frac{2(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{B}{E}\right)\left[\frac{1-\nu}{3+\nu}(R_{ci})^3 + R_{ci}(R_{co})^2\right] \tag{8}$$

$$\delta_{co} = \frac{2(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{B}{E}\right)\left[\frac{1-\nu}{3+\nu}(R_{co})^3 + (R_{ci})^2 R_{co}\right] \tag{9}$$

If $\varepsilon_x \neq 0$, we must superpose the stress system

$$\sigma_r = \sigma_t = 0 \qquad \sigma_x = E\,\varepsilon_x \tag{10 a,b,c}$$

and the corresponding additional deformation

$$u = -r\,\nu\,\varepsilon_x \tag{11}$$

Thus, we have

$$\delta_{ci} = -R_{ci}\,\nu\,\varepsilon_x \qquad \delta_{co} = -R_{co}\,\nu\,\varepsilon_x \tag{12 a,b}$$

to be added to the results given in Eqs. (8) and (9) respectively.

XLIV. THIN-WALLED CYLINDRICAL SHELLS WITH RADIAL TEMPERATURE GRADIENT

The assumption of thin wall leads to the approximation

$$\sigma_r = 0 \tag{1}$$

It is convenient to define an average temperature

$$\bar{T} = \frac{\int_{R_{ci}}^{R_{co}} T(r)\,dr}{R_{co} - R_{ci}} \tag{2}$$

Then

$$\sigma_t = \frac{\alpha E}{1-\nu} \left[\bar{T} - T(r) \right] \tag{3}$$

$$\sigma_x = \frac{\alpha E}{1-\nu} \left[\nu \bar{T} - T(r) \right] + E \varepsilon_x \tag{4}$$

This corresponds to an axial tensile force.

$$F = EA \left(\varepsilon_x - \alpha \bar{T} \right) \tag{5}$$

where A is the annular cross section area. If there is no axial force $\varepsilon_x = \alpha\bar{T}$.

XLV. THIN-WALLED LONG CYLINDRICAL SHELLS WITH TEMPERATURE GRADIENT IN AXIAL DIRECTION

It is assumed that the temperature at any point is a function of axial position only. There is no variation through the thickness of the wall. In particular, also, the following results apply to the case where a segment of thin cylinder with constant linear temperature gradient is joined to a segment of the same pipe having constant temperature. A more general situation can be treated by integrating a fourth order differential equation [34].

The general equation for symmetrical deformation of long cylindrical shell can be used for analysis [5,21,33,34].

Denoting by T the increase of the temperature of the shell from a certain uniform initial temperature and assuming that the shell is divided into infinitely thin rings by planes perpendicular to the x axis, the radial expansion of the rings due to the temperature changes is $\alpha R_c T$. This expansion can be eliminated by applying external pressure of an intensity $\frac{Et_c\alpha T}{R_c}$ which brings the shell to its initial diameter. Assume that the length b (Fig. 1) of the thin-walled cylindrical shell is long and the part of the cylindrical shell to the right of the cross section 1 - 1 has constant temperature T_∞, and to the left has a temperature which decreases linearly to the value $(T_o + T_\infty)$ at the end $x = b$.

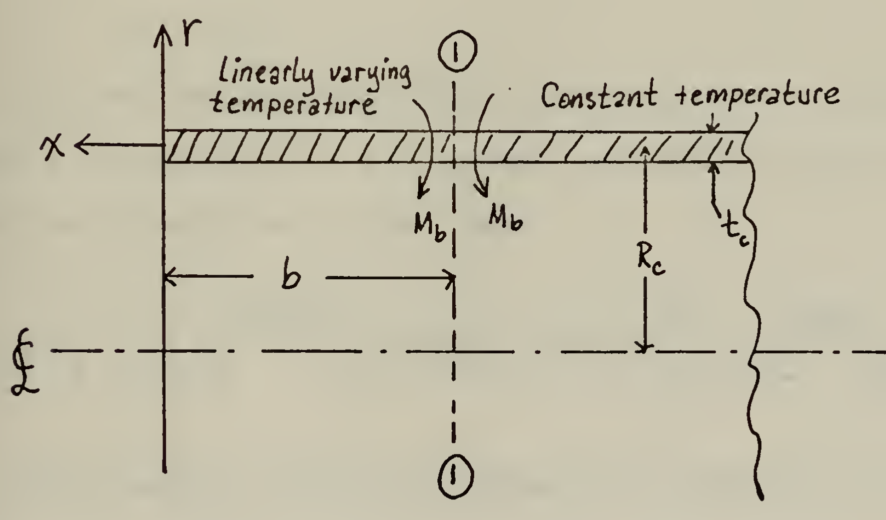


Fig. 1 Cylinder having linearly varying temperature adjacent to section having constant temperature

Then the temperature increase, as a function of x, is

$$T = \frac{T_0 x}{b} \tag{1}$$

Using this in the general differential equation and applying boundary conditions, the equations for bending moment and maximum stress at section 1 - 1 can be written, respectively, as follows

$$M_b = \beta_c D_c \frac{\alpha R_c}{2b} T_0 \tag{2}$$

$$(\sigma_x)_{max} = \frac{6 M_b}{(t_c)^2} = 0.353 \frac{E\alpha}{b} \times \sqrt{R_c t_c}\, T_0 \tag{3}$$

It has been assumed that the length $b >> R_c t_c$. Otherwise a correction to the moment equation becomes necessary and can be written as follows;

$$\Delta M_b = -M_b e^{-2\beta_c b}\left[\cos(\beta_c b) + \sin(\beta_c b)\right]^2 - 2M_b e^{-2\beta_c b}\left[\sin(\beta_c b)\right]^2 \tag{4}$$

This moment correction should be applied to the moment given by equation (2) to get the correct moment, after which

$$(\sigma_x)_{max} = \frac{6(M_b + \Delta M_b)}{(t_c)^2} \tag{5}$$

XLVI. THIN-WALLED CYLINDRICAL SHELLS UNDER INTERNAL PRESSURE, AXIAL TENSILE FORCE AND NONUNIFORM TEMPERATURE CHANGE

This section is adapted from the treatment in [2,24]. By considering the equilibrium of the element shown in Fig. 1,

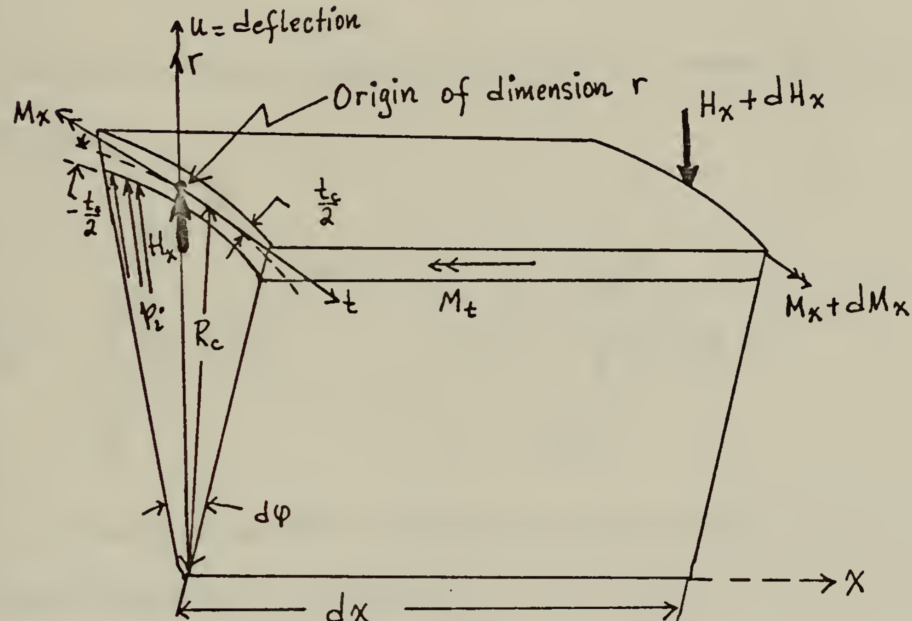


Fig. 1 Element of thin cylindrical shell showing dimensions and notations

the following equations can be written

$$q = \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \sigma_x \, dr \qquad (1)$$

$$\frac{dH_x}{dx} = p_2 - \frac{1}{R_c}\int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \sigma_t \, dr \tag{2}$$

$$H_x = \frac{dM_x}{dx} \tag{3}$$

Also the moments can be written in terms of the stresses as follows:

$$M_x = -\int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \sigma_x \, r \, dr \qquad M_t = -\int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \sigma_t \, r \, dr \tag{4 a,b}$$

By considering the geometry shown in Fig. 2, the following equations can be written

$$\varepsilon_x = e_{mx} - \left(\frac{2r}{t_c}\right) e_{bx} + \frac{\alpha r \Delta T}{t_c} + \alpha \bar{T} \tag{5}$$

$$\varepsilon_t = e_{mt} - \left(\frac{2r}{t_c}\right) e_{bt} + \frac{\alpha r \Delta T}{t_c} + \alpha \bar{T} \tag{6}$$

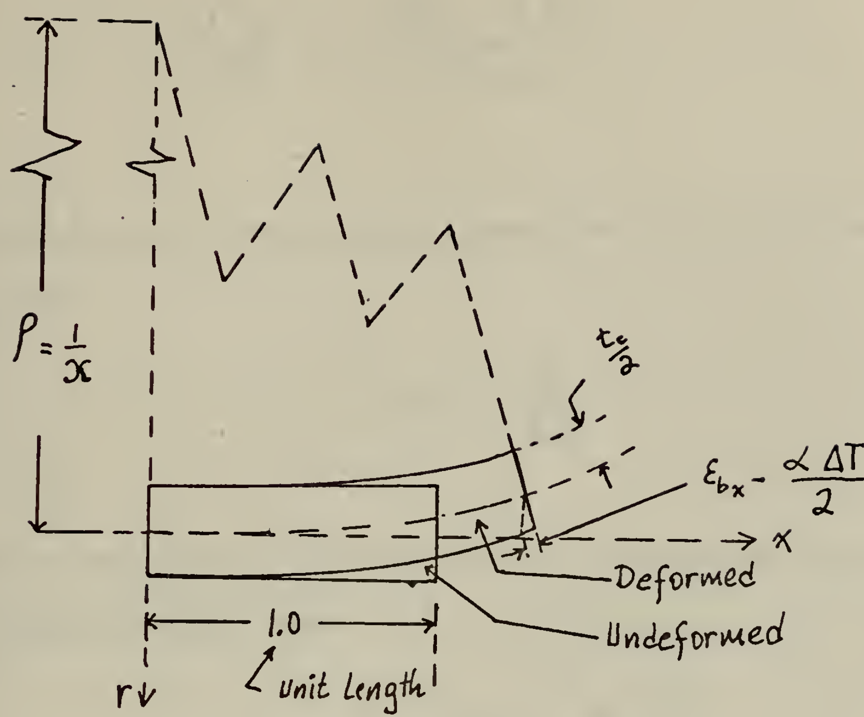


Fig. 2 Drawing showing geometry of deformed shell

where subscripts m and b denote the membrane component and bending component respectively. It has been assumed that the strain variation is linear through the wall. Also $\bar{T} = \bar{T}(x)$ denotes the local average temperature through the wall and $\Delta T = \Delta T(x)$ denotes the local excess of outside wall temperature, the variation being assumed linear.

We also have

$$\kappa = \frac{1}{\rho} = \frac{\varepsilon_{bx} - \frac{\alpha \Delta T}{2}}{\frac{t_c}{2}}$$

where κ is longitudinal curvature.

Making the customary approximation

$$\frac{d^2u}{dx^2} = \frac{2\varepsilon_{bx} - \alpha \Delta T}{t_c} \tag{7}$$

Since the cross section preserves its circularity we can write

$$\varepsilon_t = \frac{u}{R_c} \tag{8}$$

From equations (6) and (8), by comparing like powers of r, it may be seen that

$$\varepsilon_{bt} = \frac{\alpha \Delta T}{2} \tag{9}$$

$$\varepsilon_{mt} = \frac{u}{R_c} - \alpha \bar{T} \tag{10}$$

By considering the stress-strain relations, the following equations can be written

$$\sigma_x = \frac{E}{1-\nu^2}\left[\left(\varepsilon_{mx} + \nu\varepsilon_{mt}\right) - \left(\frac{2r}{t_c}\right)\left(\varepsilon_{bx} + \nu\varepsilon_{bt}\right)\right] \tag{11}$$

$$\sigma_t = \frac{E}{1-\nu^2}\left[\left(\varepsilon_{mt} + \nu\varepsilon_{mx}\right) - \left(\frac{2r}{t_c}\right)\left(\varepsilon_{bt} + \nu\varepsilon_{bx}\right)\right] \tag{12}$$

The radial stress has been neglected in eqs. (11) and (12).

Manipulation of these equations leads to the differential equation

$$\frac{d^2}{dx^2}\left(D_c\frac{d^2u}{dx^2}\right) + \frac{Et_c u}{(R_c)^2} = p_z - \frac{\nu q}{R_c} + \frac{Et_c\alpha\bar{T}}{R_c} - \frac{d^2}{dx^2}\left[\frac{E(t_c)^2\alpha\Delta T}{12(1-\nu)}\right] \tag{13}$$

For a shell with constant properties along the length, this reduces to

$$D_c\frac{d^4u}{dx^4} + \frac{Et_c u}{R_c} = p_z - \frac{\nu q}{R_c} + \frac{Et_c\alpha\bar{T}}{R_c} - \frac{E(t_c)^2\alpha}{12(1-\nu)} \times \frac{d^2}{dx^2}\Delta T \tag{14}$$

This fourth order differential equation must be solved subject to appropriate end conditions which will be discussed later in this section. Once u is known as a function of x, stresses may be found from the formulas

$$\sigma_x = \frac{q}{t_c} - \left(\frac{2r}{t_c}\right)\left[\frac{6M_x}{(t_c)^2}\right] \tag{15}$$

$$\sigma_t = \frac{Eu}{R_c} - E\alpha\bar{T} + \frac{\nu q}{t_c} - \left(\frac{2r}{t_c}\right)\left[\frac{6\nu M_x}{(t_c)^2} + \frac{E\alpha\Delta T}{2}\right] \tag{16}$$

where

$$M_x = D_c\left[\frac{d^2u}{dx^2} + \frac{(1+\nu)\alpha\,\Delta T}{t_c}\right] \tag{17}$$

The end conditions with which we may be concerned relate either to a condition at an accessible edge or to a condition at an infinite distance. At an accessible edge, any two of the following four conditions must be specified:

a) end displacement u at x = 0

b) end slope $\frac{du}{dx}$ at x = 0

c) end moment (see Eq. 17) at x = 0

d) end shear $\frac{dM_x}{dx}$ at x = 0.

At an infinite distance, we must have vanishing shear and vanishing slope, with radial displacement satisfying

$$\frac{Et_c u}{R_c} = p_i - \frac{\nu q}{R_c} + \frac{Et_c\alpha\bar{T}}{R_c} \tag{18}$$

and moment satisfying

$$M_x = \frac{D_c(1+\nu)\alpha \Delta T}{t_c} \tag{19}$$

A total of four such conditions is required to specify a unique solution to equation (14), two from each end, where the word end means either an accessible edge or an infinitely distant section.

Since such a wide variety of conditions is represented by this analysis, no attempt will be made to give specific applications, except that we may note that if the right hand side of eq. (14) is constant, we have

$$\frac{d^4 u}{dx^4} + 4\beta^4 u = Q \tag{20}$$

where β and Q are constants. The general solution of this equation is

$$u = \frac{Q}{\beta^4} + C_1 h_1(\beta x) + C_2 h_2(\beta x) + C_3 h_3(\beta x) + C_4 h_4(\beta x) \tag{21}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants to be determined by the end conditions and $h_i(\beta x)$ are functions defined and discussed in Appendix 2.

XLVII. THICK-WALLED CYLINDRICAL PRESSURE VESSELS WITH RADIAL TEMPERATURE GRADIENT

Thermal stress analysis of thick-walled cylindrical shells is extensively reported in pressure vessel literature [3,5,6, 19,21,33,34,36,42].

Analysis can be accomplished by taking an element and then applying to it equilibrium equations of statics, stress-strain relations, and compatibility. It has been assumed that the temperature is symmetrical with respect to the axis of the cylinder and constant along this axis. Therefore the deformation of the cylinder is symmetrical about its axis. Plane strain has been considered.

Equations for radial stress, circumferential stress, and deformation can be written, respectively, as follows:

$$\sigma_r = \frac{\alpha E}{(1-\nu)(r)^2}\left[\frac{(r)^2-(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\,dr - \int_{R_{ci}}^{r} T_{(r)}\, r\,dr\right] \qquad (1)$$

$$\sigma_t = \frac{\alpha E}{(1-\nu)(r)^2}\left[\frac{(r)^2+(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\,dr + \int_{R_{ci}}^{r} T_{(r)}\, r\,dr - T_{(r)}\,(r)^2\right] \qquad (2)$$

$$u = \frac{(1+\nu)\alpha}{(1-\nu)r}\left[\int_{R_{ci}}^{r} T_{(r)}\, r\,dr + \frac{(1-2\nu)(r)^2+(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\,dr\right] - r\nu\varepsilon_x \qquad (3)$$

If the axial force F_x is zero, then both ends of the cylinder are free of traction.

Equations for σ_r, σ_t, σ_x and u can be written, respectively, as follows:

$$\sigma_r = \frac{\alpha E}{(1-\nu)(r)^2}\left[\frac{(r)^2-(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\,dr - \int_{R_{ci}}^{r} T_{(r)}\, r\,dr\right] \qquad (4)$$

$$\sigma_t = \frac{\alpha E}{(1-\nu)(r)^2}\left[\frac{(r)^2+(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\,dr + \int_{R_{ci}}^{r} T_{(r)}\, r\,dr - T_{(r)}(r)^2\right] \qquad (5)$$

$$\sigma_x = \frac{\alpha E}{1-\nu}\left[\frac{2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\,dr - T_{(r)}\right] \qquad (6)$$

$$u = \frac{\alpha}{(1-\nu)r}\left[(1+\nu)\int_{R_{ci}}^{r} T_{(r)}\, r\,dr + \frac{(1-3\nu)(r)^2+(1+\nu)(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\,dr\right] \qquad (7)$$

Note that equations (4) and (5) are the same as with equations (1) and (2) respectively, because σ_r and σ_t are not dependent on σ_x.

If $\varepsilon_x = 0$, then the ends of the cylinder are not free of traction. Equations for σ_r, σ_t, σ_x and u can be written, respectively, as follows:

$$\sigma_r = \frac{\alpha E}{(1-\nu)(r)^2}\left[\frac{(r)^2-(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\, dr - \int_{R_{ci}}^{r} T_{(r)}\, r\, dr\right] \tag{8}$$

$$\sigma_t = \frac{\alpha E}{(1-\nu)(r)^2}\left[\frac{(r)^2+(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\, dr + \int_{R_{ci}}^{r} T_{(r)}\, r\, dr - T_{(r)}\,(r)^2\right] \tag{9}$$

$$\sigma_x = \frac{\alpha E}{(1-\nu)}\left[\frac{2\nu}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\, dr - T_{(r)}\right] \tag{10}$$

$$u = \frac{(1+\nu)\alpha}{(1-\nu)\alpha}\left[\int_{R_{ci}}^{r} T_{(r)}\, r\, dr + \frac{(1-2\nu)(r)^2+(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\int_{R_{ci}}^{R_{co}} T_{(r)}\, r\, dr\right] \tag{11}$$

Again the stresses σ_r and σ_t are not changed.

If the distribution of temperature T(r) as a function of radial position is known, the integrals in all above equations can be calculated; then the stresses and deformation may be obtained at any desired position.

XLVIII. THICK-WALLED CYLINDRICAL PRESSURE VESSELS WITH LOGARITHMIC RADIAL TEMPERATURE GRADIENT

Steady state radial heat flow through the walls of a pressure vessel, an obviously important special case, leads to the so-called logarithmic gradient given by

$$T_{(r)} = T_i \cdot \frac{\log_e\left(\frac{R_{co}}{r}\right)}{\log_e\left(\frac{R_{co}}{R_{ci}}\right)} \tag{1}$$

where T_i represents the temperature of the inside surface and where, without essential loss of generality, the temperature of the outside surface is taken to be zero.

Using this particular temperature distribution in the formulas of the preceeding section leads to those given in the present section.

If the total axial force is zero and the ends of the cylinder are free of traction, equations for σ_r, σ_t, σ_z and u can be written, respectively, as follows:

$$\sigma_r = \frac{\alpha E T_i}{2(1-\nu)\log_e\left(\frac{R_{co}}{R_{ci}}\right)}\left\{-\log_e\left(\frac{R_{co}}{r}\right) - \frac{(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\left[1-\frac{(R_{co})^2}{(r)^2}\right]\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right\} \tag{2}$$

$$\sigma_t = \frac{\alpha E T_i}{2(1-\nu)\log_e\left(\frac{R_{co}}{R_{ci}}\right)}\left\{1-\log_e\left(\frac{R_{co}}{r}\right) - \frac{(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\left[1+\frac{(R_{co})^2}{(r)^2}\right]\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right\} \tag{3}$$

$$\sigma_x = \frac{\alpha E T_i}{2(1-\nu)\log_e\left(\frac{R_{co}}{R_{ci}}\right)}\left\{1 - 2\log_e\left(\frac{R_{co}}{r}\right) - \left[\frac{2(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\right]\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right\} \tag{4}$$

$$u = \frac{\alpha T_i}{4(1-\nu)r\log_e\left(\frac{R_{co}}{R_{ci}}\right)}\left\{(1+\nu)(r)^2\left[1+2\log_e\left(\frac{R_{co}}{r}\right)\right]-(1+\nu)(R_{ci})^2\left[1+2\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right]+\left[(1-3\nu)(r)^2+(1+\nu)(R_{ci})^2\right]\left[1-\frac{\quad}{(R_{co})^2-(R_{ci})^2}\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right]\right\} \tag{5}$$

The maximum circumferential stress, which is of greatest interest, occurs at the inner surface [1-a].

$$(\sigma_t)_{max} = \frac{E\alpha T_i}{2(1-\nu)\log_e\left(\frac{R_{co}}{R_{ci}}\right)}\left[1-\frac{2(R_{co})^2}{(R_{co})^2-(R_{ci})^2}\cdot\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right] \tag{6}$$

If ε_x is zero and the ends of cylinder are not free of traction, equations for σ_r, σ_t, σ_x and u can be written, respectively, as follows:

σ_r = The same as (2) (7)

σ_t = The same as (3) (8)

$$\sigma_x = \frac{\alpha E T_i}{2(1-\nu)\log_e\left(\frac{R_{co}}{R_{ci}}\right)}\left\{\nu - 2\log_e\left(\frac{R_{co}}{R_{ci}}\right) - \left[\frac{2\nu(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\right]\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right\} \tag{9}$$

$$F = \frac{\pi\alpha E T_i}{2(1-\nu)\log_e\left(\frac{R_{co}}{R_{ci}}\right)}\left\{\left[(R_{co})^2-(R_{ci})^2\right]\left[\nu - 2\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right] - 2\nu(R_{ci})^2\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right\} \tag{10}$$

$$u = \frac{\alpha T_i}{4(1-\nu)r\log_e\left(\frac{R_{co}}{R_{ci}}\right)}\left\{(1+\nu)(r)^2\left[1+2\log_e\left(\frac{R_{co}}{r}\right)\right] - (1+\nu)(R_{ci})^2\left[1+2\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right] + \left[(1+\nu)(1-2\nu)(r)^2+(1+\nu)(R_{ci})^2\right]\left[1-\frac{2(R_{ci})^2}{(R_{co})^2-(R_{ci})^2}\log_e\left(\frac{R_{co}}{R_{ci}}\right)\right]\right\} \tag{11}$$

Maximum circumferential stress, which occurs at the inner surface, is given by Equation (6) above.

XLVIX. THICK-WALLED SPHERICAL PRESSURE VESSELS WITH STEADY STATE TEMPERATURE GRADIENT

The well known [36] radial and circumferential stresses and deformation equations can be written, respectively, as follows:

$$\sigma_r = \frac{2\alpha E}{(1-\nu)(r)^3}\left\{\left[\frac{(r)^3-(R_{ci})^3}{(R_{co})^3-(R_{ci})^3}\right]\int_{R_{ci}}^{R_{co}} T_{(r)}(r)^2 dr - \int_{R_{ci}}^{r} T_{(r)}(r)^2 dr\right\} \tag{1}$$

$$\sigma_t = \frac{2\alpha E}{(1-\nu)(r)^3}\left\{\frac{2(R_{co})^3+(R_{ci})^3}{2\left[(R_{co})^3-(R_{ci})^3\right]}\int_{R_{ci}}^{R_{co}} T_{(r)}(r)^2 dr + \frac{1}{2}\int_{R_{ci}}^{r} T_{(r)}(r)^2 dr - \frac{1}{2}T_{(r)}\right\} \tag{2}$$

$$u = \frac{\alpha}{(1-\nu)(r)^2}\left\{(1+\nu)\int_{R_{ci}}^{r} T_{(r)}(r)^2 dr + \left[\frac{2(1-2\nu)(r)^3+(1+\nu)(R_{ci})^3}{(R_{co})^3-(R_{ci})^3}\right]\int_{R_{ci}}^{R_{co}} T_{(r)}(r)^2 dr\right\} \tag{3}$$

In the case of steady state heat flow the temperature at any distance from the center can be written as

$$T_{(r)} = T_{i}\cdot\frac{R_{ci}}{R_{co}-R_{ci}}\left(\frac{R_{co}}{r}-1\right) \tag{4}$$

where we take $T(R_{co}) = 0$.

Substituting equation (4) into the equations (1), (2), and (3) gives:

$$\sigma_r = \frac{\alpha E T_i}{(1-\nu)} \cdot \frac{R_{ci} R_{co}}{(R_{co})^3 - (R_{ci})^3} \left\{ R_{ci} + R_{co} - \frac{1}{r}\left[(R_{co})^2 + R_{ci} R_{co} + (R_{ci})^2 \right] + \frac{(R_{ci})^2 (R_{co})^2}{(r)^3} \right\} \tag{5}$$

$$\sigma_t = \frac{\alpha E T_i}{(1-\nu)} \times \frac{R_{ci} R_{co}}{(R_{co})^3 - (R_{ci})^3} \left\{ R_{ci} + R_{co} - \frac{1}{2r}\left[(R_{co})^2 + R_{ci} R_{co} + (R_{ci})^2 \right] - \frac{(R_{ci})^2 (R_{co})^2}{2(r)^3} \right\} \tag{6}$$

$$u = \frac{\alpha T_i}{(1-\nu)(r)^2} \left\{ (1+\nu)\left[\frac{R_{ci} R_{co}}{2} \times \frac{(r)^2 - (R_{ci})^2}{R_{co} - R_{ci}} - \frac{R_{ci}}{3} \times \frac{(r)^3 - (R_{ci})^3}{R_{co} - R_{ci}} \right] + \left[2(1-2\nu)(r)^3 + (1+\nu)(R_{ci})^2 \right]\left[\frac{R_{ci} R_{co}}{2} \times \frac{R_{co} + R_{ci}}{(R_{co})^3 - (R_{ci})^3} - \frac{R_{ci}}{3} \times \frac{1}{R_{co} - R_{ci}} \right] \right\} \tag{7}$$

L. <u>THICK-WALLED TWO-LAYER CYLINDRICAL PRESSURE VESSELS WITH DIFFERENT MATERIALS UNDER SHRINK-FIT TOGETHER WITH INTERNAL AND EXTERNAL PRESSURE</u>

Although the general case of multilayered vessels having n layers is given in Appendix 3, the analysis there involves some manipulation. Since the two-layered vessel represents the most important application, formulas for this particular case are presented here. (See also discussion at end of Appendix 3.)

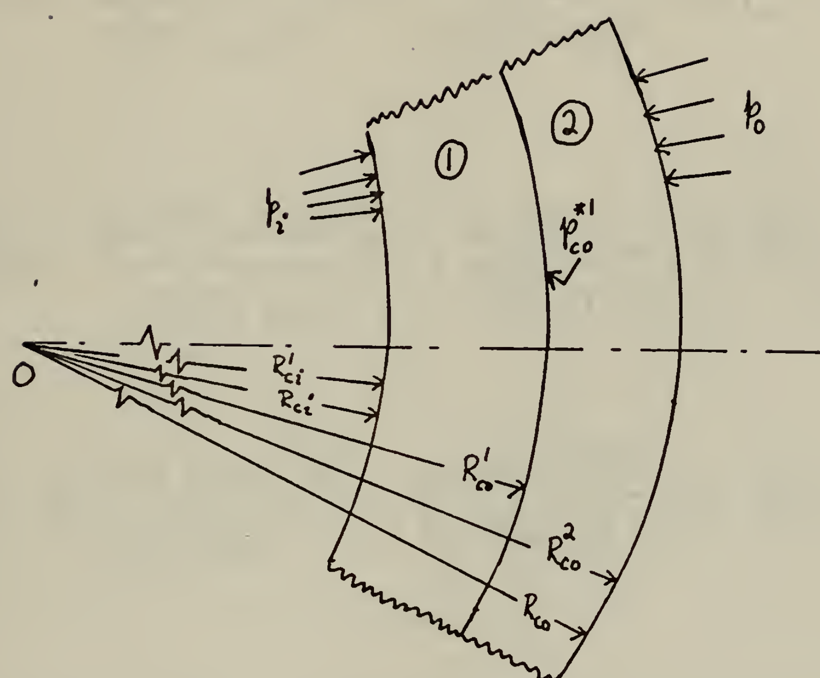


Fig. 1 Dimensions and notation for two-layered vessel

The interface pressure can be written as follows

$$p_{co}^{*1} = \frac{\dfrac{\delta_1^*}{R'_{co}} + \varepsilon_x(\nu_2 - \nu_1) + \dfrac{2(1-\nu_1^2)(R_{ci})^2\, p_i}{E_1\left[(R'_{co})^2 - (R_{ci})^2\right]} + \dfrac{2(1-\nu_2^2)(R_{co})^2\, p_o}{E_2\left[(R_{co})^2 - (R'_{co})^2\right]}}{\dfrac{1}{E_1}\left\{(1-\nu_1^2)\dfrac{\left[(R'_{co})^2 + (R_{ci})^2\right]}{\left[(R'_{co})^2 - (R_{ci})^2\right]} - \nu_1 - \nu_1^2\right\} + \dfrac{1}{E_2}\left\{(1-\nu_2^2)\dfrac{\left[(R_{co})^2 + (R'_{co})^2\right]}{\left[(R_{co})^2 - (R'_{co})^2\right]} + \nu_2 + \nu_2^2\right\}} \qquad (1)$$

For the inner cylinder, equations for stresses and deformation can be written as follows:

$$\sigma_r^1 = \frac{(R_{ci})^2 p_i - (R'_{co})^2 p_{co}^{*1}}{(R'_{co})^2 - (R_{ci})^2} - \frac{(R_{ci})^2 (R'_{co})^2}{(R'_{co})^2 - (R_{ci})^2} \left(p_i - p_{co}^{*1} \right) \frac{1}{(r)^2} \tag{2}$$

$$\sigma_t^1 = \frac{(R_{ci})^2 p_i - (R'_{co})^2 p_{co}^{*1}}{(R'_{co})^2 - (R_{ci})^2} + \frac{(R_{ci})^2 (R'_{co})^2}{(R'_{co})^2 - (R_{ci})^2} \left(p_i - p_{co}^{*1} \right) \frac{1}{(r)^2} \tag{3}$$

$$u_1 = \frac{1-\nu_1}{E_1} \cdot \frac{(R_{ci})^2 p_i - (R'_{co})^2 p_{co}^{*1}}{(R'_{co})^2 - (R_{ci})^2} \cdot r + \frac{1+\nu_1}{E_1} \cdot \frac{(R_{ci})^2 (R'_{co})^2}{(R'_{co})^2 - (R_{ci})^2} \left(p_i - p_{co}^{*1} \right) \frac{1}{r} - \frac{\nu_1}{E_1} \sigma_x^1 r \tag{4}$$

For the outer, cylinder equations for stresses and deformation can be written as follows:

$$\sigma_r^2 = \frac{(R'_{co})^2 p_{co}^{*1} - (R_{co})^2 p_o}{(R_{co})^2 - (R'_{co})^2} + \frac{(R'_{co})^2 (R_{co})^2}{(R_{co})^2 - (R'_{co})^2} \left(p_{co}^{*1} - p_o \right) \frac{1}{(r)^2} \tag{5}$$

$$\sigma_t^2 = \frac{(R'_{co})^2 p_{co}^{*1} - (R_{co})^2 p_o}{(R_{co})^2 - (R'_{co})^2} - \frac{(R'_{co})^2 (R_{co})^2}{(R_{co})^2 - (R'_{co})^2} \left(p_{co}^{*1} - p_o \right) \frac{1}{(r)^2} \tag{6}$$

$$u_2 = \frac{1-\nu_2}{E_2} \cdot \frac{(R'_{co})^2 p_{co}^{*1} - (R_{co})^2 p_o}{(R_{co})^2 - (R'_{co})^2} \cdot r + \frac{1+\nu_2}{E_2} \cdot \frac{(R'_{co})^2 (R_{co})^2}{(R_{co})^2 - (R'_{co})^2} \left(p_{co}^{*1} - p_o \right) \frac{1}{r} - \frac{\nu_2}{E_2} \sigma_x^2 r \tag{7}$$

LI. THICK-WALLED TWO-LAYER CYLINDRICAL PRESSURE VESSELS WITH SAME MATERIALS UNDER SHRINK-FIT TOGETHER WITH INTERNAL AND EXTERNAL PRESSURE

The interface pressure can be written as follows:

$$p_{co}^{*1} = \frac{\dfrac{\delta_1^*}{R_{co}'} + \dfrac{2(1-\nu^2)}{E}\left\{\dfrac{(R_{ci})^2\, p_i}{\left[(R_{co}')^2-(R_{ci})^2\right]} + \dfrac{(R_{co})^2\, p_o}{\left[(R_{co})^2-(R_{co}')^2\right]}\right\}}{\dfrac{1-\nu^2}{E}\left\{\dfrac{\left[(R_{co}')^2+(R_{ci})^2\right]}{\left[(R_{co}')^2-(R_{ci})^2\right]} + \dfrac{\left[(R_{co})^2+(R_{co}')^2\right]}{\left[(R_{co})^2-(R_{co}')^2\right]}\right\}} \qquad (1)$$

For the inner cylinder equations for stresses and deformation can be written as follows:

$$\sigma_r^1 = \frac{(R_{ci})^2 p_i - (R_{co}')^2 p_{co}^{*1}}{(R_{co}')^2-(R_{ci})^2} - \frac{(R_{ci})^2(R_{co}')^2}{(R_{co}')^2-(R_{ci})^2}\left(p_i - p_{co}^{*1}\right)\frac{1}{(r)^2} \qquad (2)$$

$$\sigma_t^1 = \frac{(R_{ci})^2 p_i - (R_{co}')^2 p_{co}^{*1}}{(R_{co}')^2-(R_{ci})^2} + \frac{(R_{ci})^2(R_{co}')^2}{(R_{co}')^2-(R_{ci})^2}\left(p_i - p_{co}^{*1}\right)\frac{1}{(r)^2} \qquad (3)$$

$$u_1 = \frac{1-\nu}{E}\cdot\frac{(R_{ci})^2 p_i - (R'_{co})^2 p^{*1}_{co}}{(R'_{co})^2 - (R_{ci})^2}\cdot r + \frac{1+\nu}{E}\cdot\frac{(R_{ci})^2 (R'_{co})^2}{(R'_{co})^2 - (R_{ci})^2}(p_i - p^{*1}_{co})\frac{1}{r} - \frac{\nu}{E}\sigma_x^1 r \qquad (4)$$

For outer cylinder equations for stresses and deformation can be written as follows:

$$\sigma_r^2 = \frac{(R'_{co})^2 p^{*1}_{co} - (R_{co})^2 p_o}{(R_{co})^2 - (R'_{co})^2} + \frac{(R'_{co})^2 (R_{co})^2}{(R_{co})^2 - (R'_{co})^2}(p^{*1}_{co} - p_o)\frac{1}{(r)^2} \qquad (5)$$

$$\sigma_t^2 = \frac{(R'_{co})^2 p^{*1}_{co} - (R_{co})^2 p_o}{(R_{co})^2 - (R'_{co})^2} - \frac{(R'_{co})^2 (R_{co})^2}{(R_{co})^2 - (R'_{co})^2}(p^{*1}_{co} - p_o)\frac{1}{(r)^2} \qquad (6)$$

$$u_2 = \frac{1-\nu}{E}\cdot\frac{(R'_{co})^2 p^{*1}_{co} - (R_{co})^2 p_o}{(R_{co})^2 - (R'_{co})^2}\cdot r + \frac{1+\nu}{E}\cdot\frac{(R'_{co})^2 (R_{co})^2}{(R_{co})^2 - (R'_{co})^2}(p^{*1}_{co} - p_o)\frac{1}{r} - \frac{\nu}{E}\sigma_x^2 r \qquad (7)$$

Equations for the cases with internal and external pressure only can be obtained by putting $p_o = 0$ and $p_i = 0$, respectively, into the equations given in Section L and in this section.

LII. OPTIMAL DESIGN OF MULTILAYER PRESSURE VESSELS OF SAME MATERIAL

Before presenting the sequence of formulas, a remark is in order about what is meant by "optimizing" in this case. It means simply that the overall volume or weight of the metallic material is minimized, subject to the conditions that the inside radius is the given number R_{ci}, that the internal pressure is the given number p_i, and that in the loaded condition each of the shells is stressed up to the allowable limit, but no higher (Fig. 1)

It has been presumed that the design pressure p_i, has been specified, and that the material selection has provided values of E, Young's modulus, of ν, Poisson's ratio, and of S_t, the allowable tensile stress. It should also be explicitly stated that all shells are presumed to be composed of the same material, or, at least, that the values of S_t, E, and ν are the same for all shells. Also, one should be aware that the following formulas yield results which are precisely valid only at some distance from any "end effects". The case of plane strain is postulated [1].

Equations applicable to the choice of the maximum shearing stress theory of failure will be denoted as (MSS) and equations applicable to the choice of the maximum normal stress theory of failure will be denoted (MNS).

If the MNS theory is used, S_t is simply the allowable normal or tensile stress; if (MSS) theory is used, S_t is twice the allowable shearing stress.

Note: Symbols α, β, κ, ρ, n used herin are defined by equations herein. These special meanings, used only in this section, are not reflected in the Notations, p. 7.

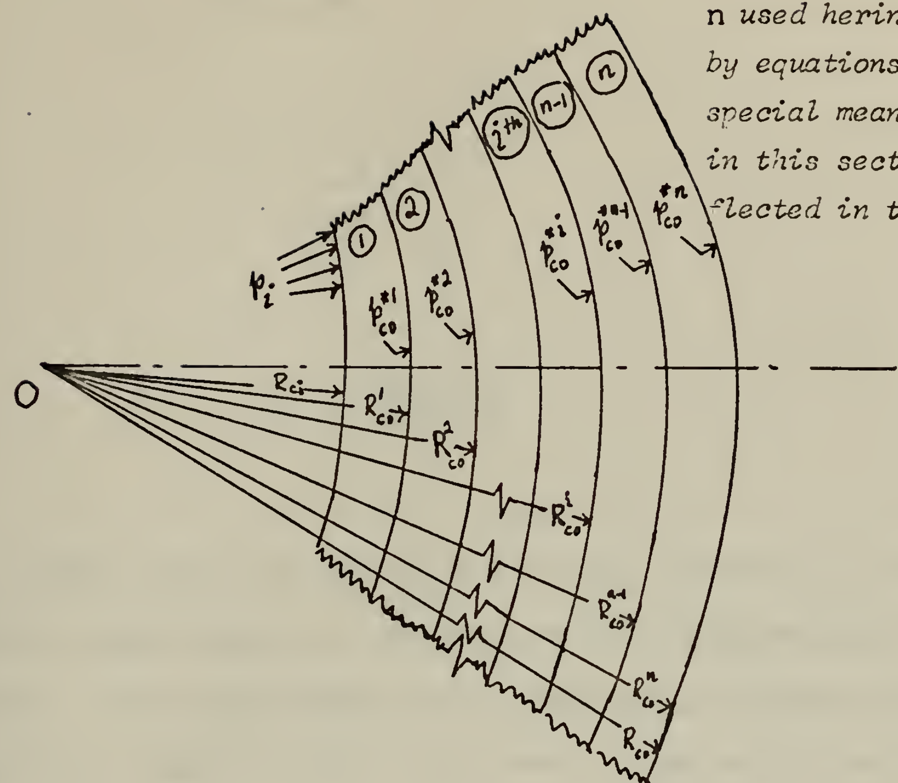


Fig. 1 Dimensions and notation for multi-layer vessel.

The sequence of design formulas can be written as follows:

$$\rho = \frac{p_{io}}{S_t} \tag{1}$$

$$\alpha = \frac{1}{1 - \frac{2\rho}{n}} \qquad \text{(2) MSS}$$

$$\alpha = (1+\rho)^{\frac{1}{n}} \qquad \text{(2) MNS}$$

$$\beta = \alpha \qquad \text{(3) MSS}$$

$$\beta = \frac{1}{\left(\frac{2}{\alpha} - 1\right)} \qquad \text{(3) MNS}$$

$$K = (\beta)^n - 1 \qquad (4)$$

where K is the ratio of metallic volume to the internal volume, and thus is a measure of the "optimality" of the design. For both theories of failure, optimum design results in shell radii which are in geometric sequence.

$$R_{co}^{i} = \alpha (\beta)^{\frac{i}{2}} \qquad (5)$$

The radial compression which exist in the fully assembled and pressurized condition between i^{th} and $(i+1)^{th}$ shells is

$$Z_i = (n-i) \frac{p_i}{n} \qquad \text{(6) MSS}$$

$$Z_i = S_t\left[(\alpha)^{n-i} - 1\right] \qquad \text{(6) MNS}$$

The desired radial interferences are given by

$$\delta_i = \frac{2(1-\nu^2) R_{co}^{i} p_{co}^{*i}}{nE} \qquad \text{(7) MSS}$$

$$\delta_i = \frac{(1-\nu^2)(S_t + Z_i) R_{co}^{i} (\beta+1)}{(\beta+1) E} \qquad \text{(7) MNS}$$

These radial interferences are the amount by which the outer radius of the i^{th} shell exceeds the inner radius of the $(i+1)^{th}$ shell when the shells have been manufactured and are at the same temperature and before the assembly process has started.

The sequence of formulas for stress can be written as follows.

These formulas relate to the problem of calculating stresses at various points of the assembled vessel in the unloaded and in the pressurized condition.

Hoop stress:

$$(\sigma_t)_\ell^i = \left(\frac{p_{co}^{*i}}{n}\right)\left[\frac{2\beta}{\beta-1} - (n+i)\right] \qquad \text{(8) MSS}$$

$$(\sigma_t)_\ell^i = S_t \qquad \text{(8) MNS}$$

$$\left(\sigma_t\right)_s^{i} = \left(\frac{p_{CD}^{*i}}{n}\right)\left[\frac{2}{\beta-1} - (n+i)\right] \qquad \text{(9) MSS}$$

$$\left(\sigma_t\right)_s^{i} = \frac{2(\alpha Z_{i-1} - \beta Z_i)}{\alpha(\beta-1)} \qquad \text{(9) MNS}$$

Axial stresses:

$$\sigma_x^{i} = \left[\frac{1}{\beta-1} - \frac{n}{(\beta)^n-1} - (n+i)\right]\left[2\nu \cdot \frac{p_{CD}^{*1}}{n}\right] \qquad \text{(10) MSS}$$

$$\sigma_x^{i} = \left[\frac{\alpha(\beta)^n - \beta(\alpha)^n}{\beta-\alpha} - \frac{(\alpha)^n - \alpha}{\alpha-1}\right]\left[\frac{(1+\beta-2\alpha)\nu S_t}{(S_t)^n-1}\right] \qquad \text{(10) MNS}$$

$$\sigma_x^{i} = \sigma_x^{i+1} - \frac{2\nu\, p_{CD}^{*1}}{n} \qquad \text{(11) MSS}$$

$$\sigma_x^{i} = \sigma_x^{i+1} - \nu S_t(1+\beta-2\alpha)(\alpha)^{n-i} \qquad \text{(11) MNS}$$

It should be noted that the formula (10) MNS gives only the single value $\sigma_x^{\ n}$; and to get other values of $\sigma_x^{\ i}$ for the (MNS) case, formula (11) MNS should be used.

To complete the calculations of stresses, the formulas will be exhibited which show the changes in going from the unpressurized to the fully pressurized condition.

$$(\Delta\sigma_t)_\ell^i = (\Delta\sigma_t)_s^i = \frac{p_{co}^{*1}\left[(\beta)^{n-i}+1\right]}{\left[(\beta)^n-1\right]} \tag{12}$$

$$\Delta Z_i = \frac{p_{co}^{*1}\left[(\beta)^{n-i}-1\right]}{\left[(\beta)^n-1\right]} \tag{13}$$

$$\Delta\sigma_x^i = \frac{p_{co}^{*1} + \dfrac{F}{\pi(R_{ci})^2}}{\left[(\beta)^n-1\right]} \tag{14}$$

APPENDIX 1

SEMI-INFINITE THIN-WALLED CYLINDRICAL SHELLS WITHOUT CLOSURE UNDER END LOADING

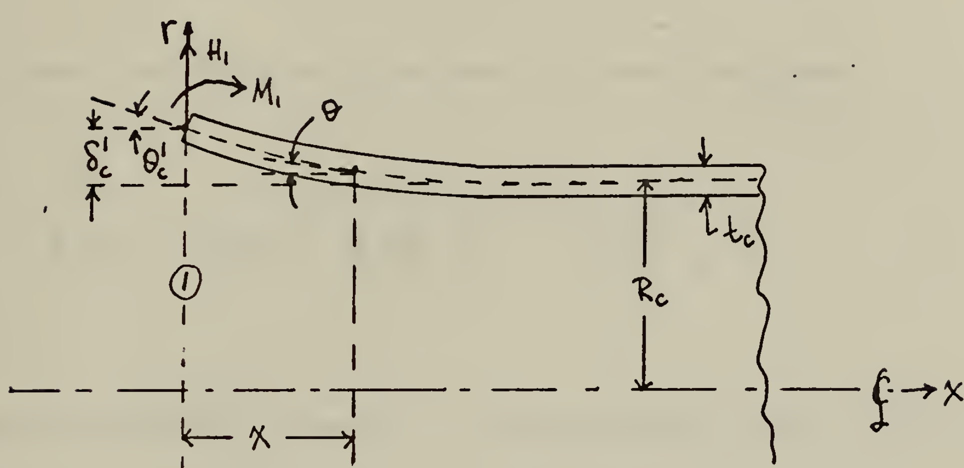


Fig. 1 End region of semi-infinite cylinder

The solution of the governing fourth order differential equation, which also satisfies the requirement that the radial displacement, u , and its derivatives vanish as $x \to \infty$, may be written as (cf. Section XLVI)

$$u = A_1 f_1(\beta_c x) + A_2 f_2(\beta_c x) \qquad (1)$$

where A_1 and A_2 are arbitrary constants and $f_1(\beta_c s)$ and $f_2(\beta_c x)$ are any two of the four functions

$$\Lambda(\beta_c x) = e^{-\beta_c x}\cos(\beta_c x) \qquad \zeta(\beta_c x) = e^{-\beta_c x}\sin(\beta_c x)$$

$$\Omega(\beta_c x) = \Lambda(\beta_c x) + \zeta(\beta_c x) \qquad \Psi(\beta_c x) = \Lambda(\beta_c x) - \zeta(\beta_c x) \tag{2}$$

which have the following values for zero argument

$$\Lambda(0) = \Omega(0) = \Psi(0) = 1 \qquad \zeta(0) = 0 \tag{3}$$

and which have the schedule of derivatives shown in Table-1.

OPERATION FUNCTION

$\frac{d^0}{dx^0}$	$\Omega(\beta_c x)$	$\Psi(\beta_c x)$	$\Lambda(\beta_c x)$	$\zeta(\beta_c x)$
$\frac{d^1}{dx^1}$	$-2\beta_c \zeta(\beta_c x)$	$-2\beta_c \Lambda(\beta_c x)$	$-\beta_c \Omega(\beta_c x)$	$\beta_c \Psi(\beta_c x)$
$\frac{d^2}{dx^2}$	$-2(\beta_c)^2 \Psi(\beta_c x)$	$2(\beta_c)^2 \Omega(\beta_c x)$	$2(\beta_c)^2 \zeta(\beta_c x)$	$-2(\beta_c)^2 \Lambda(\beta_c x)$
$\frac{d^3}{dx^3}$	$4(\beta_c)^3 \Lambda(\beta_c x)$	$-4(\beta_c)^3 \zeta(\beta_c x)$	$2(\beta_c)^3 \Psi(\beta_c x)$	$2(\beta_c)^3 \Omega(\beta_c x)$
$\frac{d^4}{dx^4}$	$-4(\beta_c)^4 \Omega(\beta_c x)$	$-4(\beta_c)^4 \Psi(\beta_c x)$	$4(\beta_c)^4 \Lambda(\beta_c x)$	$-4(\beta_c)^4 \zeta(\beta_c x)$

Table-1 SCHEDULE OF DERIVATIVES FOR FUNCTIONS (2)

A convenient selection gives (u) in terms of the edge shear H_1 and moment M_1. (See Fig. 1), viz.

$$u = \frac{\left[H_1 \Lambda(\beta_c x) + \beta_c M_1 \Psi(\beta_c x)\right]}{2(\beta_c)^3 D_c} \tag{4}$$

This gives the following evaluations of (u) and its derivatives at the accessible end x = 0

$$u = \frac{H_1 + \beta_c M_1}{2(\beta_c)^3 D_c}$$

$$\frac{du}{dx} = - \frac{H_1 + 2\beta_c M_1}{2(\beta_c)^2 D_c}$$

$$\frac{d^2u}{dx^2} = \frac{M_1}{D_c} \qquad \text{at} \quad x = 0 \tag{5}$$

$$\frac{d^3u}{dx^3} = \frac{H_1}{D_c}$$

These results provide the influence coefficients frequently encountered in the body herof; e.g. cf. Equation (3) of Section XXV.

APPENDIX 2

FINITE THIN-WALLED CYLINDRICAL SHELLS WITHOUT CLOSURE UNDER END LOADING

In this appendix we discuss the behavior of finite thin-walled cylinder under edge (end) loadings and obtain the appropriate corresponding influence coefficients.

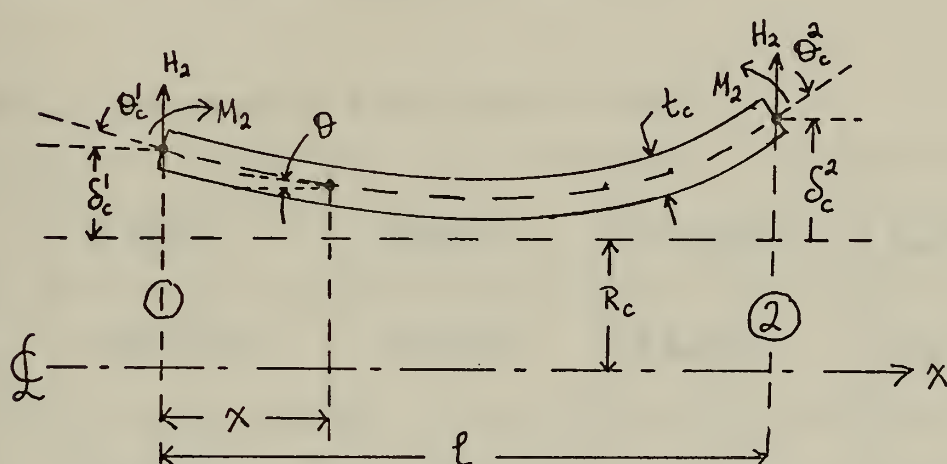


Fig. 1 Finite cylindrical shell

The functions

$$
\begin{aligned}
h_1(\beta_c x) &= \cosh(\beta_c x)\cos(\beta_c x) \\
h_2(\beta_c x) &= \frac{1}{2}\left[\cosh(\beta_c x)\sin(\beta_c x) + \sinh(\beta_c x)\cos(\beta_c x)\right] \\
h_3(\beta_c x) &= \frac{1}{2}\sinh(\beta_c x)\sin(\beta_c x) \\
h_4(\beta_c x) &= \frac{1}{4}\left[\cosh(\beta_c x)\sin(\beta_c x) - \sinh(\beta_c x)\cos(\beta_c x)\right]
\end{aligned}
\tag{1}
$$

provide four linearly independent solutions to the homogeneous form of Equation (2) of Section XLVI, the solution of which can thus be written

$$u = a_1 h_1(\beta_c x) + a_2 h_2(\beta_c x) + a_3 h_3(\beta_c x) + a_4 h_4(\beta_c x) \tag{2}$$

These functions have the following values for zero argument

$$h_1(0) = 1 \qquad h_2(0) = h_3(0) = h_4(0) = 0 \tag{3}$$

and their derivatives are shown in Table 1

	$h_1(\beta_c x)$	$h_2(\beta_c x)$	$h_3(\beta_c x)$	$h_4(\beta_c x)$
$\frac{d}{dx}$	$-4\beta_c h_4(\beta_c x)$	$\beta_c h_1(\beta_c x)$	$\beta_c h_2(\beta_c x)$	$\beta_c h_3(\beta_c x)$
$\frac{d^2}{dx^2}$	$-4(\beta_c)^2 h_3(\beta_c x)$	$-4(\beta_c)^2 h_4(\beta_c x)$	$(\beta_c)^2 h_1(\beta_c x)$	$(\beta_c)^2 h_2(\beta_c x)$
$\frac{d^3}{dx^3}$	$-4(\beta_c)^3 h_2(\beta_c x)$	$-4(\beta_c)^3 h_3(\beta_c x)$	$-4(\beta_c)^3 h_4(\beta_c x)$	$(\beta_c)^3 h_1(\beta_c x)$
$\frac{d^4}{dx^4}$	$-4(\beta_c)^4 h_1(\beta_c x)$	$-4(\beta_c)^4 h_2(\beta_c x)$	$-4(\beta_c)^4 h_3(\beta_c x)$	$-4(\beta_c)^4 h_4(\beta_c x)$

Table-1 SCHEDULE OF DERIVATIVES OF THE FUNCTIONS h_i

Thus we obtain,

$$\frac{du}{dx} = -4\beta_c a_1 h_4(\beta_c x) + \beta_c a_2 h_1(\beta_c x) + \beta_c a_3 h_2(\beta_c x) + \beta_c a_4 h_3(\beta_c x) \tag{4}$$

$$\frac{d^2u}{dx^2} = -4(\beta_c)^2 a_1 h_3(\beta_c x) - 4(\beta_c)^2 a_2 h_4(\beta_c x) + (\beta_c)^2 a_3 h_1(\beta_c x) + (\beta_c)^2 a_4 h_2(\beta_c x) \tag{5}$$

$$\frac{d^3u}{dx^3} = -4(\beta_c)^3 a_1 h_2(\beta_c x) - 4(\beta_c)^3 a_2 h_3(\beta_c x) - 4(\beta_c)^3 a_3 h_4(\beta_c x) + (\beta_c)^3 a_4 h_1(\beta_c x) \tag{6}$$

From Equations (3) it is easy to obtain the evaluations at the left end (indicated by superscript 1)

$$\delta_c^1 = a_1 \tag{7}$$

$$\theta_c^1 = \beta_c a_2 \tag{8}$$

$$\frac{M_1}{(\beta_c)^2 D_c} = a_3 \tag{9}$$

$$\frac{H_1}{(\beta_c)^3 D_c} = a_4 \tag{10}$$

Similarly, equations (2), (4), (5), and (6) can be rewritten for the end 2 as follows:

$$\delta_c^2 = a_1 h_1^2 + a_2 h_2^2 + a_3 h_3^2 + a_4 h_4^2 \tag{11}$$

$$\frac{d\delta_c^2}{dx} = -4\beta_c a_1 h_4^2 + \beta_c a_2 h_1^2 + \beta_c a_3 h_2^2 + \beta_c a_4 h_3^2 \tag{12}$$

$$\frac{d^2\delta_c^2}{dx^2} = -4(\beta_c)^2 a_1 h_3^2 - 4(\beta_c)^2 a_2 h_4^2 + (\beta_c)^2 a_3 h_1^2 + (\beta_c)^2 a_4 h_2^2 = \frac{M_2}{D_c} \tag{13}$$

$$\frac{d^3\delta_c^2}{dx^3} = -4(\beta_c)^3 a_1 h_2^2 - 4(\beta_c)^3 a_2 h_3^2 - 4(\beta_c)^3 a_3 h_4^2 + (\beta_c)^3 a_4 h_1^2 = -\frac{H_2}{D_c} \tag{14}$$

where

$$h_1^2 = h_1(\beta_c \ell) \quad h_2^2 = h_2(\beta_c \ell) \quad h_3^2 = h_3(\beta_c \ell) \quad h_4^2 = h_4(\beta_c \ell) \tag{15}$$

(Note use of superscript 2 to indicate evaluation at $x = \ell$.)

Equations (9) and (10) give coefficients a_3 and a_4 in terms of the end loadings. Thus, the only unknown quantities in equations (13) and (14) are the coefficients a_1 and a_2, and simultaneous solution of (13) and (14) gives

$$a_1 = \frac{\beta_c M_1\left[h_1^2 h_3^2 + 4(h_4^2)^2\right] + H_1\left[h_2^2 h_3^3 - h_1^2 h_4^2\right] - \beta_c M_2 h_3^2 - H_2 h_4^2}{4(\beta_c)^3 D_c\left[(h_3^2)^2 - h_1^2 h_4^2\right]} \tag{16}$$

$$a_2 = \frac{-\beta_c M_1\left[h_1^2 h_2^2 + 4 h_3^2 h_4^2\right] + H_1\left[h_1^2 h_3^2 - (h_2^2)^2\right] + \beta_c M_2 h_2^2 + H_2 h_3^2}{4(\beta_c)^3 D_c\left[(h_3^2)^2 - h_1^2 h_4^2\right]} \tag{17}$$

We emphasize again the use of superscript 2 to indicate evaluation at the right end; exponents appear outside parentheses.

These expressions for a_1 and a_2 could be rewritten in terms of hyperbolic and trigonometric functions of $(\beta_c \ell)$ and then simplified somewhat, but actually, the expressions given here are as compact as any alternate form.

We are thus finally in a position to write the influence coefficients in the expressions

$$\delta_c^1 = \delta_c^{1M_1} M_1 + \delta_c^{1H_1} H_1 + \delta_c^{1M_2} M_2 + \delta_c^{1H_2} H_2 \tag{18}$$

$$\theta_c^1 = \theta_c^{1M_1} M_1 + \theta_c^{1H_1} H_1 + \theta_c^{1M_2} M_2 + \theta_c^{1H_2} H_2 \tag{19}$$

$$\delta_c^2 = \delta_c^{2M_1} M_1 + \delta_c^{2H_1} H_1 + \delta_c^{2M_2} M_2 + \delta_c^{2H_2} H_2 \tag{20}$$

$$\theta_c^2 = \theta_c^{2M_1} M_1 + \theta_c^{2H_1} H_1 + \theta_c^{2M_2} M_2 + \theta_c^{2H_2} H_2 \tag{21}$$

and we obtain the evaluations

$$\delta_c^{1M_1} = \frac{\beta_c\left[h_1^2 h_3^2 + 4(h_4^2)^2\right]}{X} \tag{22}$$

$$\theta_c^{1M_1} = \frac{(\beta_c)^2\left[h_1^2 h_2^2 + 4 h_3^2 h_4^2\right]}{X} \tag{26}$$

$$\delta_c^{1H_1} = \frac{h_2^2 h_3^2 - h_1^2 h_4^2}{X} \tag{23}$$

$$\theta_c^{1H_1} = -\frac{\beta_c\left[h_1^2 h_3^2 - (h_2^2)^2\right]}{X} \tag{27}$$

$$\delta_c^{1M_2} = -\frac{\beta_c h_3^2}{X} \tag{24}$$

$$\theta_c^{1M_2} = -\frac{(\beta_c)^2 h_2^2}{X} \tag{28}$$

$$\delta_c^{1H_2} = -\frac{h_4^2}{X} \tag{25}$$

$$\theta_c^{1H_2} = -\frac{\beta_c h_3^2}{X} \tag{29}$$

where $$X = 4(\beta_c)^3 D_c\left[(h_3)^2 - h_1^2 h_4^2\right] \tag{30}$$

$$\delta_c^{2M_1} = \delta_c^{1M_1} h_1^2 - \theta_c^{1M_1}\frac{h_2^2}{\beta_c} + \frac{h_3^2}{(\beta_c)^2 D_c} \tag{31}$$

$$\delta_c^{2H_1} = \delta_c^{1H_1} h_1^2 - \theta_c^{1H_1} \frac{h_3^2}{\beta_c} + \frac{h_4^2}{(\beta_c)^3 D_c} \tag{32}$$

$$\delta_c^{2M_2} = \delta_c^{1M_2} h_1^2 - \theta_c^{1M_2} \frac{h_2^2}{\beta_c} \tag{33}$$

$$\delta_c^{2H_2} = \delta_c^{1H_2} h_1^2 - \theta_c^{1H_2} \frac{h_2^2}{\beta_c} + \frac{h_2^2}{\beta_c D_c} \tag{34}$$

$$\theta_c^{2M_1} = -4\beta_c \delta_c^{1M_1} h_4^2 - \theta_c^{1M_1} h_1^2 + \frac{h_3^2}{(\beta_c)^2 D_c} \tag{35}$$

$$\theta_c^{2H_1} = -4\beta_c \delta_c^{1H_1} h_4^2 - \theta_c^{1H_1} h_1^2 \tag{36}$$

$$\theta_c^{2M_2} = -4\beta_c \delta_c^{1M_2} h_4^2 - \theta_c^{1M_2} h_1^2 \tag{37}$$

$$\theta_c^{2H_2} = -4\beta_c \delta_c^{1H_2} h_4^2 - \theta_c^{1H_2} h_1^2 \tag{38.}$$

APPENDIX 3

SHRINK ASSEMBLED MULTILAYER CYLINDRICAL VESSELS UNDER PRESSURIZATION

(This appendix was contributed by Prof. J. E. Brock. His notation has been changed so as to conform with that appearing elsewhere herein.)

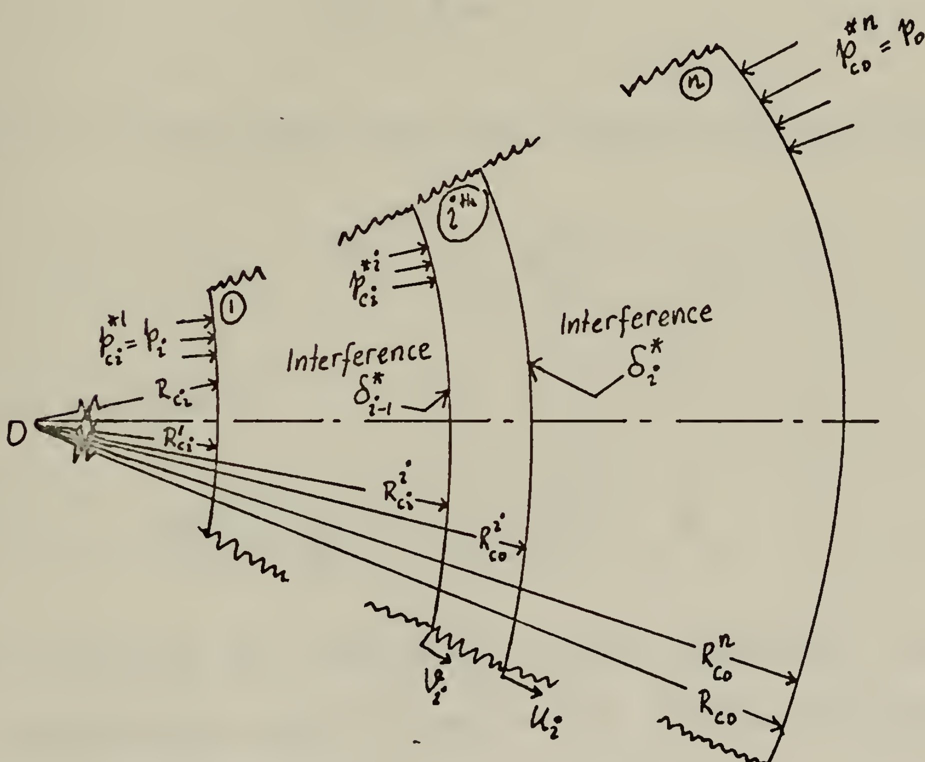


Fig. 1 Dimensions and notations for shrink assembled multilayer vessel.

Figure 1 indicates the configuration to be analyzed. The radii are prior to assembly and pressurization. The interferences

$$\delta_{i} = R_{co}^{i} - R_{ci}^{i+1} \tag{1}$$

are small compared to either radius on the right. Accordingly in the analysis which follows we may use the approximation

$$R_{ci}^{i} = R_{co}^{i-1} \tag{2}$$

Also, it is clear that interface pressures are related by

$$p_{ci}^{*i+1} = p_{co}^{*i} \tag{3}$$

In particular

$$p_{ci}^{*1} = p_{i} \qquad\qquad p_{co}^{*n} = p_{o} \tag{4}$$

the internally and externally applied pressures, respectively.

By applying Lame's formulas (cf. Section IX hereof) to the i^{th} shell, we find

$$(\sigma_r)_{r=R_{co}^{i-1}} = -p_{co}^{*i-1} \tag{5}$$

$$(\sigma_r)_{r=R_{co}^{i}} = -p_{co}^{*i} \tag{6}$$

$$(\sigma_r)_{r=R_{co}^{i-1}} = \frac{\left[(R_{co}^{i})^2+(R_{co}^{i-1})^2\right]p_{co}^{*i-1} - 2\,(R_{co}^{i})^2\,p_{co}^{*i}}{(R_{co}^{i})^2-(R_{co}^{i-1})^2} \tag{7}$$

$$(\sigma_r)_{r=R_{co}^{i}} = \frac{2(R_{co}^{i-1})^2\,p_{co}^{*i-1} - \left[(R_{co}^{i})^2+(R_{co}^{i-1})^2\right]p_{co}^{*i}}{(R_{co}^{i})^2-(R_{co}^{i-1})^2} \tag{8}$$

Assuming a state of plane strain with ε_x = const. for all layers, fundamental equations of elasticity lead to

$$\sigma_x^{i} = E_i\,\varepsilon_x + 2\nu_i\,\frac{\left[(R_{co}^{i-1})^2 p_{co}^{*i-1} - (R_{co}^{i})^2\,p_{co}^{*i}\right]}{\left[(R_{co}^{i})^2-(R_{co}^{i-1})^2\right]} \tag{9}$$

Note that this is constant throughout the i^{th} shell.

$$u_i = A_i\,p_{co}^{*i-1} + B_i\,p_{co}^{*i} - \nu_i\,R_{co}^{i}\,\varepsilon_x \tag{10}$$

is the outward deformation at the outside (see Fig. 1) and

$$v = C_i \, p_{co}^{*i-1} + D_i \, p_{co}^{*i} - \nu_i \, R_{co}^{i-1} \, \varepsilon_x \tag{11}$$

is the outward deformation at the inside. The constants are

$$A_i = \frac{2(1-\nu_i^2) \, R_{co}^{i} \, (R_{co}^{i-1})^2}{E_i \left[(R_{co}^{i})^2 - (R_{co}^{i-1})^2 \right]} \tag{12}$$

$$B_i = \frac{R_{co}^{i}}{E_i} \left\{ (\nu_i + \nu_i^2) - (1-\nu_i^2) \frac{\left[(R_{co}^{i})^2 + (R_{co}^{i-1})^2 \right]}{\left[(R_{co}^{i})^2 - (R_{co}^{i-1})^2 \right]} \right\} \tag{13}$$

$$C_i = \frac{R_{co}^{i-1}}{E_i} \left\{ (\nu_i + \nu_i^2) + (1-\nu_i^2) \frac{\left[(R_{co}^{i})^2 + (R_{co}^{i-1})^2 \right]}{\left[(R_{co}^{i})^2 - (R_{co}^{i-1})^2 \right]} \right\} \tag{14}$$

$$D_i = - \frac{2(1-\nu_i^2) \, R_{co}^{i-1} \, (R_{co}^{i})^2}{E_i \left[(R_{co}^{i})^2 - (R_{co}^{i-1})^2 \right]} \tag{15}$$

By presuming that there are no gaps between shells under any conditions, geometrical continuity leads to the relation

$$v_{i+1} = u_i + \delta_i^* \qquad i = (1, 2, \ldots, n-1) \tag{16}$$

Substituting equations (10) and (11) into (16) gives

$$D_{i+1}\, p_{co}^{*\,i+1} + \left(C_{i+1} - B_i\right) p_{co}^{*\,i} - A_i\, p_{co}^{*\,i-1} = \delta_i^{*} + \varepsilon_x R_{co}^{i} \left(V_{i+1} - V_i\right) \tag{17}$$

$$i = (1, 2, \ldots, n-1)$$

Transposing known quantities to the right and assembling the results in matrix form, we have the system

$$[S]\{P\} = \{\Delta\} + \varepsilon_x \{V\} + \{L\} \tag{18}$$

where

$$[S] = \begin{bmatrix} (C_n - B_{n-1}) & -A_{n-1} & 0 & & & & \\ D_{n-1} & (C_{n-1} - B_{n-2}) & -A_{n-2} & & & & \\ 0 & D_{n-2} & (C_{n-2} - B_{n-3}) & \ddots & & & \\ & & & & (C_4 - B_3) & -A_3 & 0 \\ & & & & D_3 & (C_3 - B_2) & -A_2 \\ & & & & 0 & D_2 & (C_2 - B_1) \end{bmatrix}$$

is a triple band square matrix.

$$\{P\} = \left\{ p_{co}^{*n-1},\ p_{co}^{*n-2},\ \ldots,\ p_{co}^{*2},\ p_{co}^{*1} \right\} \tag{19}$$

is a column matrix of unknown interface pressures.

$$\{\Delta\} = \left\{ \delta_{n-1}^{*},\ \delta_{n-2}^{*},\ \ldots,\ \delta_{2}^{*},\ \delta_{1}^{*} \right\} \tag{20}$$

is a column matrix of the radial interferences,

$$\{V\} = \left\{ R_{co}^{n-1}(\nu_n - \nu_{n-1}),\ R_{co}^{n-2}(\nu_{n-1} - \nu_{n-2}),\ \ldots,\ R_{co}^{2}(\nu_3 - \nu_2),\ R_{co}^{1}(\nu_2 - \nu_1) \right\} \tag{21}$$

is a column matrix containing differences between Poisson's ratios, and

$$\{L\} = \left\{ -D_n p_o,\ 0,\ 0,\ \ldots,\ 0,\ A_1 p_2 \cdot \right\} \tag{22}$$

is a column matrix involving exterior and interior fluid pressures.

By use of equation (9), we can find the total axial tensile force F as

$$F = \pi \sum_{i'=1}^{n} \left[(R_{co}^{i'})^2 - (R_{co}^{i'-1})^2 \right] \sigma_x^{i'}$$

$$= G \varepsilon_x + k + 2\pi \langle M \rangle \{P\} \tag{23}$$

where

$$G = \pi \sum \left[(R_{co}^{i'})^2 - (R_{co}^{i'-1})^2 \right] E_{i'} \tag{24}$$

$$k = 2\pi \left[\nu_1 (R_{ci})^2 p_i - \nu_n (R_{co})^2 p_o \right] \tag{25}$$

and

$$\langle M \rangle = \langle (\nu_n - \nu_{n-1})(R_{co}^{n-1})^2, (\nu_{n-1} - \nu_{n-2})(R_{co}^{n-2})^2, \ldots, (\nu_3 - \nu_2)(R_{co}^{2})^2, (\nu_2 - \nu_1)(R_{co}^{1})^2 \rangle \tag{26}$$

is a row matrix.

Equation (23) permits expressing ε_x in terms of F as follows

$$\varepsilon_x = \frac{F - k - 2\pi \langle M \rangle \{P\}}{G} \tag{27}$$

If ε_x is given, there is no difficulty in solving equation (18) and the solution may be written

$$\{P\} = [C]\{\Delta\} + \varepsilon_x [C]\{V\} + [C]\{L\} \tag{28}$$

where

$$[C] = [S]^{-1} \tag{29}$$

Having the interface pressures in {P} , it is elementary to find stresses σ_r and σ_t in the individual layers. Equation (9) gives the axial stresses, and equation (23) gives the total axial force.

If the net total tensile force F is given, equation (27) is used to replace ε_x on the right side of equation (18), which can then be written

$$[S^*]\{P\} = \{\Delta\} + F\{V^*\} + \{L^*\} \tag{30}$$

where

$$[S^*] = [S] + \frac{2\pi}{G}\{V\}\langle M\rangle \tag{31}$$

is a square matrix which, however, is no longer banded,

$$\{V^*\} = \frac{\{V\}}{G} \tag{32}$$

and

$$[L^*] = \{L\} - \frac{k}{G}\{V\} \tag{33}$$

The solution is

$$\{P\} = [C^*]\{\Delta\} + F[C^*]\{V^*\} + [C^*]\{L^*\} \tag{34}$$

where

$$[C^*] = [S^*]^{-1} \tag{35}$$

The interface pressures are now known and ε_x may be obtained from equation (27). Thus the stresses may easily be calculated. It should be remarked that this plane strain analysis gives results which are significantly different from results obtained by analyzing the assembly and loading of thin <u>rings</u>. The latter analysis is sometimes incorrectly given as being applicable to the case of multilayer cylinders. It should also be noted that the present analysis assumes that ε_x, from initial unpressurized unassembled condition to final pressurized assembled condition, is the same for each cylinder. Practical methods of assembly usually imply adding one layer at a time and during the successive addition of layers to achieve the final assembled vessel, it may not be possible to assure that ε_x is indeed the same for each layer. However, after assembly is complete, the additional ε_x due to pressurization is indeed the same for each layer.

The special case n = 2 is of particular interest and leads to the following evaluation if ε_x is specified.

$$p_{co}^{*1} = \frac{\dfrac{\delta_1^*}{R'_{co}} + \varepsilon_x(\nu_2 - \nu_1) + \dfrac{2(1-\nu^2)(R_{ci})^2 p_i}{E_1\left[(R'_{co})^2 - (R_{ci})^2\right]} + \dfrac{2(1-\nu^2)(R_{co})^2 p_o}{E_2\left[(R_{co})^2 - (R'_{co})^2\right]}}{\dfrac{1}{E_1}\left\{(1-\nu_1^2)\dfrac{\left[(R'_{co})^2 + (R_{ci})^2\right]}{\left[(R'_{co})^2 - (R_{ci})^2\right]} - \nu_1 - \nu_1^2\right\} + \dfrac{1}{E_2}\left\{(1-\nu_2^2)\dfrac{\left[(R_{co})^2 + (R'_{co})^2\right]}{\left[(R_{co})^2 - (R'_{co})^2\right]} + \nu_2 + \nu_2^2\right\}} \tag{36}$$

This may be compared with results given in Section L.

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13. ABSTRACT

Formulas relating to the elastic behavior of axisymmetrically loaded cylindrical and spherical pressure vessels have been collected from various sources. Their presentation is unified by the employment of a uniform system of notation. Loadings include interior and exterior pressurization, radial and axial temperature gradients, axial loads, and centrifugal force fields. Some cases of buckling are treated, and shrink assembled multilayer vessels are included. A large bibliography is listed.

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